On Solvency, Model Uncertainty and Risk Measures

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Outline

Fundamental problem in Finance/Insurance

- Risk factors: $\mathbf{X} = (X_1, \ldots, X_d)$
- Model assumption: $X_i \sim F_i$, F_i known, $i = 1, \ldots, d$
- A financial position $\Psi(\mathbf{X})$
- A risk measure/pricing function: $\rho(\Psi(\mathbf{X}))$

Calculate $\rho(\Psi(\mathbf{X}))$

Calculating $\rho(\Psi(\mathbf{X}))$

Example:

- $\Psi(\mathbf{X}) = \sum_{i=1}^{d} X_i$
- $\rho = \mathrm{VaR}_p$ or $\rho = \mathrm{ES}_p$

Challenge:

- We need a *joint* model for the random vector $\mathbf{X}$
- Joint models are hard to get by

We will focus on the above special choices of Ψ and ρ.

VaR and ES

$\mathrm{VaR}_p(X)$

For $p \in (0,1)$,

$$\mathrm{VaR}_p(X) = F_X^{-1}(p) = \inf\{x \in \mathbb{R} : F_X(x) \geq p\}$$

$\mathrm{ES}_p(X)$

For $p \in (0,1)$,

$$\mathrm{ES}_p(X) = \frac{1}{1-p}\int_p^1 \mathrm{VaR}_q(X)\mathrm{d}q \underset{(F \text{ cont.})}{=} \mathbb{E}\left[X | X > \mathrm{VaR}_p(X)\right]$$

VaR and ES

A related quantity Left-tail-ES:

$\mathrm{LES}_p(X)$

For $p \in (0,1)$,

$$\mathrm{LES}_p(X) = \frac{1}{p}\int_0^p \mathrm{VaR}_q(X)\mathrm{d}q = -\mathrm{ES}_{1-p}(-X)$$

Fréchet problem

Denote

$$\mathcal{S}_d = \mathcal{S}_d(F_1, \ldots, F_d) = \left\{ \sum_{i=1}^{d} X_i : X_i \sim F_i,\ i = 1, \ldots, d \right\}$$

- Every element in $\mathcal{S}_d$ is a possible risk position.
- Determination of $\mathcal{S}_d$: very challenging.
  - Think about $\mathcal{S}_2(U[0,1], U[0,1])$ ... open question!

Worst- and best-values of VaR and ES

The Fréchet (unconstrained) problems for VaR_p

$$\overline{\mathrm{VaR}}_p(S_d) = \sup\{\mathrm{VaR}_p(S) : S \in \mathcal{S}_d(F_1, \ldots, F_d)\},$$

$$\underline{\mathrm{VaR}}_p(S_d) = \inf\{\mathrm{VaR}_p(S) : S \in \mathcal{S}_d(F_1, \ldots, F_d)\}.$$

Same notation for ES_p and LES_p.

Worst- and best-values of VaR and ES

- ES is subadditive:

$$\overline{\mathrm{ES}}_p(S_d) = \sum_{i=1}^{d} \mathrm{ES}_p(X_i).$$

Similarly $\underline{\mathrm{LES}}_p(S_d) = \sum_{i=1}^{d} \mathrm{LES}_p(X_i)$.

- $\overline{\mathrm{VaR}}_p(S_d)$, $\underline{\mathrm{VaR}}_p(S_d)$ and $\underline{\mathrm{ES}}_p(S_d)$: generally open questions

Challenge for $\underline{\mathrm{ES}}_p(S_d)$

To calculate $\underline{\mathrm{ES}}_p(S_d)$ one naturally seeks a safest risk in $\mathcal{S}_d$.

Mathematical difficulty

Common understanding of the most dangerous scenario:

- Comonotonicity - well accepted notion

Understanding concerning the safest scenario:

- $d = 2$: counter-monotonicity
- $d \geq 3$: question mark! (?!)
  - Calls for notions of extremal negative dependence.

Mathematical difficulty

ES respects convex order: the natural order of risk preference.

Convex order

We write $X \leq_{cx} Y$ if $\mathbb{E}[f(X)] \leq \mathbb{E}[f(Y)]$ for all convex functions f such that the two expectations exist.

Finding $\underline{\mathrm{ES}}_p(S_d)$

Search for a smallest element in $\mathcal{S}_d$ with respect to convex order, if it exists.

Mathematical difficulty

VaR does not respect convex order: more tricky

- Good news: the questions for $\overline{\mathrm{VaR}}_p(S_d)$, $\underline{\mathrm{VaR}}_p(S_d)$ and $\underline{\mathrm{ES}}_p(S_d)$ are mathematically similar.

Finding $\overline{\mathrm{VaR}}_p(S_d)$

Search for a smallest element in $\mathcal{S}_d(\hat{F}_1, \ldots, \hat{F}_d)$ with respect to convex order, where $\hat{F}_i$ is the p-tail-conditional distribution of F_i.

- $\underline{\mathrm{VaR}}_p(S_d)$ is symmetric to $\overline{\mathrm{VaR}}_p(S_d)$.

Summary of existing results

$d = 2$:

- fully solved analytically

$d \geq 3$:

- Homogeneous model $(F_1 = \cdots = F_d)$
  - $\underline{\mathrm{ES}}_p(S_d)$ solved analytically for decreasing densities, e.g. Pareto, Exponential
  - $\overline{\mathrm{VaR}}_p(S_d)$ solved analytically for tail-decreasing densities, e.g. Pareto, Gamma, Log-normal
- Inhomogeneous model
  - Few analytical results: current research
- Numerical methods available: Rearrangement Algorithm

VaR bounds

$d = 2$, Makarov (1981) and Rüschendorf (1982)

For any $p \in (0,1)$,

$$\overline{\mathrm{VaR}}_p(S_2) = \inf_{x \in [0,1-p]} \{F_1^{-1}(p+x) + F_2^{-1}(1-x)\},$$

and

$$\underline{\mathrm{VaR}}_p(S_2) = \sup_{x \in [0,p]} \{F_1^{-1}(x) + F_2^{-1}(p-x)\}.$$

- A large outcome is coupled with a small outcome.

VaR bounds - homogeneous model

Sharp VaR bounds (Wang, Peng and Yang, 2013)

Suppose that the density function of F is decreasing on $[b, \infty)$ for some $b \in \mathbb{R}$. Then, for $p \in [F(b), 1)$, and $X \overset{d}{\sim} F$,

$$\overline{\mathrm{VaR}}_p(S_d) = d\mathbb{E}[X|X \in [F^{-1}(p+(d-1)c), F^{-1}(1-c)]],$$

where c is the smallest number in $[0, \frac{1}{d}(1-p)]$ such that

$$\int_{p+(d-1)c}^{1-c} F^{-1}(t)\mathrm{d}t \geq \frac{1-p-dc}{d}((d-1)F^{-1}(p+(d-1)c) + F^{-1}(1-c)).$$

Red part clearly has an ES-type form.

- $c = 0$: $\overline{\mathrm{VaR}}_p(S_d) = \overline{\mathrm{ES}}_p(S_d)$.

VaR bounds - homogeneous model

Sharp VaR bounds II

Suppose that the density function of F is decreasing on its support. Then for $p \in (0,1)$ and $X \overset{d}{\sim} F$,

$$\underline{\mathrm{VaR}}_p(S_d) = \max\{(d-1)F^{-1}(0) + F^{-1}(p), d\mathbb{E}[X|X \le F^{-1}(p)]\}.$$

Red part has an LES form.

ES bounds - homogeneous model

Sharp ES bounds (Bernard, Jiang and Wang, 2014)

Suppose that the density function of F is decreasing on its support. Then for $p \in (1-dc, 1)$, $q = (1-p)/d$ and $X \overset{d}{\sim} F$,

$$\begin{aligned}\underline{\mathrm{ES}}_p(S_d) &= \frac{1}{q}\int_0^q \Big((d-1)F^{-1}((d-1)t) + F^{-1}(1-t)\Big)\,\mathrm{d}t,\\ &= (d-1)^2\mathrm{LES}_{(d-1)q}(X) + \mathrm{ES}_{1-q}(X),\end{aligned}$$

where c is the smallest number in $[0, \frac{1}{d}]$ such that

$$\int_{(d-1)c}^{1-c} F^{-1}(t)\mathrm{d}t \geq \frac{1-dc}{d}((d-1)F^{-1}((d-1)c) + F^{-1}(1-c)).$$

- One large outcome is coupled with $d-1$ small outcomes.

Complete mixability

The homogeneous VaR and ES bounds are based on the notion of complete mixability:

Complete mixability, Wang and Wang (2011)

A distribution function F on $\mathbb{R}$ is called d-completely mixable (d-CM) if there exist d random variables $X_1, \dots, X_d \sim F$ such that

$$\mathbb{P}(X_1 + \cdots + X_d = dk) = 1,$$

for some $k \in \mathbb{R}$.

- Equivalently, $\mathcal{S}_d(F, \dots, F)$ contains a constant.

Complete mixability

- Some examples of d-CM distributions for all $d \geq 2$: Normal, Student t, Cauchy, Uniform.
- Most relevant result: F has a monotone density on a finite interval with a mean condition (depends on d) is d-CM.
    - Examples: (truncated) Pareto, Gamma, Log-normal.
- Inhomogeneous version called joint mixability.
- A full characterization of these classses is at the moment is widely open.

Numerical calculation

Rearrangement Algorithm (RA): Embrecths, Puccetti and Rüschendorf (2013).

- A fast numerical procedure
- Based on the CM-idea
- Discretization of relevant quantile regions
- d possibly large
- Applicable to $\overline{\mathrm{VaR}}_p$, $\underline{\mathrm{VaR}}_p$ and $\underline{\mathrm{ES}}_p$

Asymptotic equivalence

Consider the case $d \to \infty$. What would happen to $\overline{\mathrm{VaR}}_p(S_d)$?

- Clearly always $\overline{\mathrm{VaR}}_p(S_d) \le \overline{\mathrm{ES}}_p(S_d)$.
- Recall that $\overline{\mathrm{VaR}}_p(S_d)$ has an ES-type part.

Under some weak conditions,

$$\lim_{d\to\infty} \frac{\overline{\mathrm{ES}}_p(S_d)}{\overline{\mathrm{VaR}}_p(S_d)} = 1.$$

This was shown first for homogeneous models and then extended to general inhomogeneous models.

Asymptotic equivalence - homogeneous model

Theorem 1

In the homogeneous model, $F_1 = F_2 = \cdots = F$, for $p \in (0,1)$ and $X \sim F$, we have that

$$\lim_{d\to\infty} \frac{1}{d} \overline{\mathrm{VaR}}_p(S_d) = \mathrm{ES}_p(X).$$

- Similar limits hold for a large class of risk measures

Asymptotic equivalence - worst-cases

Theorem 2 (Embrechts, Wang and Wang, 2014)

Suppose the continuous distributions F_i, $i \in \mathbb{N}$ satisfy that for $X_i \sim F_i$ and some $p \in (0,1)$,

(i) $\mathbb{E}[|X_i - \mathbb{E}[X_i]|^k]$ *is uniformly bounded for some* $k > 1$;

(ii) $\liminf_{d\to\infty} \frac{1}{d} \sum_{i=1}^{d} \mathrm{ES}_p(X_i) > 0$.

Then as $d \to \infty$,

$$\frac{\overline{\mathrm{ES}}_p(S_d)}{\overline{\mathrm{VaR}}_p(S_d)} = 1 + O(d^{1/k-1}).$$

Asymptotic equivalence - best-cases

Similar results holds for $\underline{\mathrm{VaR}}_p$ and $\underline{\mathrm{ES}}_p$: assume (i) and

(iii) $\liminf_{d\to\infty} \frac{1}{d}\sum_{i=1}^{d} \mathrm{LES}_p(X_i) > 0,$

then

$$\lim_{d\to\infty} \frac{\underline{\mathrm{VaR}}_p(S_d)}{\underline{\mathrm{LES}}_p(S_d)} = 1,$$

$$\lim_{d\to\infty} \frac{\underline{\mathrm{ES}}_p(S_d)}{\sum_{i=1}^{d} \mathbb{E}[X_i]} = 1,$$

and

$$\frac{\underline{\mathrm{VaR}}_p(S_d)}{\underline{\mathrm{ES}}_p(S_d)} \approx \frac{\sum_{i=1}^{d} \mathrm{LES}_p(X_i)}{\sum_{i=1}^{d} \mathbb{E}[X_i]} \leq 1, \quad d \to \infty.$$

Example: Pareto(2) risks

Bounds on VaR and ES for the sum of d Pareto(2) distributed rvs for $p = 0.999$; VaR_p^+ corresponds to the comonotonic case.

	$d = 8$	$d = 56$
$\underline{\mathrm{VaR}}_p$	31	53
$\underline{\mathrm{ES}}_p$	178	472
VaR_p^+	245	1715
$\overline{\mathrm{VaR}}_p$	465	3454
$\overline{\mathrm{ES}}_p$	498	3486
$\overline{\mathrm{VaR}}_p / \mathrm{VaR}_p^+$	1.898	2.014
$\overline{\mathrm{ES}}_p / \overline{\mathrm{VaR}}_p$	1.071	1.009

Example: Pareto(θ) risks

Bounds on the VaR and ES for the sum of $d = 8$ Pareto(θ)-distributed rvs for $p = 0.999$.

	$\theta = 1.5$	$\theta = 2$	$\theta = 3$	$\theta = 5$	$\theta = 10$
$\overline{\mathrm{VaR}}_p$	1897	465	110	31.65	9.72
$\overline{\mathrm{ES}}_p$	2392	498	112	31.81	9.73
$\overline{\mathrm{ES}}_p/\overline{\mathrm{VaR}}_p$	1.261	1.071	1.018	1.005	1.001

Dependence-uncertainty spread

Theorem 3 (Embrechts, Wang and Wang, 2014)

Take $1 > q \geq p > 0$. *Suppose that the continuous distributions* $F_i,\ i \in \mathbb{N}$, *satisfy (i) and (iii), and* $\limsup_{d\to\infty} \frac{\sum_{i=1}^d \mathbb{E}[X_i]}{\sum_{i=1}^d \mathrm{ES}_p(X_i)} < 1$, *then*

$$\liminf_{d\to\infty} \frac{\overline{\mathrm{VaR}}_q(S_d) - \underline{\mathrm{VaR}}_q(S_d)}{\overline{\mathrm{ES}}_p(S_d) - \underline{\mathrm{ES}}_p(S_d)} \geq 1.$$

- The uncertainty spread of VaR is generally bigger than that of ES.
- In recent Consultative Documents of the Basel Committee, $\mathrm{VaR}_{0.99}$ is compared with $\mathrm{ES}_{0.975}$: $p = 0.975$ and $q = 0.99$.

Dependence-uncertainty spread

ES and VaR of $S_d = X_1 + \cdots + X_d$, where

- $X_i \sim \text{Pareto}(2 + 0.1i),\ i = 1, \ldots, 5;$
- $X_i \sim \text{Exp}(i - 5),\ i = 6, \ldots, 10;$
- $X_i \sim \text{Log–Normal}(0, (0.1(i - 10))^2),\ i = 11, \ldots, 20.$

	$d = 5$			$d = 20$		
	best	worst	spread	best	worst	spread
$\text{ES}_{0.975}$	22.48	44.88	22.40	29.15	102.35	73.20
$\text{VaR}_{0.975}$	9.79	41.46	31.67	21.44	100.65	79.21
$\text{VaR}_{0.9875}$	12.06	56.21	44.16	22.12	126.63	104.51
$\text{VaR}_{0.99}$	12.96	62.01	49.05	22.29	136.30	114.01
$\frac{\overline{\text{ES}}_{0.975}}{\overline{\text{VaR}}_{0.975}}$		1.08			1.02	

Challenges

Open mathematical questions:

- Characterization of complete and joint mixability
- Characterization of $\mathcal{S}_d$
- Find $\overline{\mathrm{VaR}}_p$ under more general settings, especially in the inhomogeneous model
- Partial dependence information and realistic scenarios
- Marginal uncertainty and statistical estimation
- Many more ...

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THANK YOU!