Life Insurance Purchasing to Reach a Bequest

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Given:

A constant force of interest r.

A random variable T (future lifetime of (x)) which has

hazard function $\lambda(t)$

survival function $s(t)$

pdf $g(t)$.

A loading factor Θ, such that insurance premiums are calculated using a force of interest of $(1 + \Theta)\lambda$.

Individual will buy term insurance, the amount of which can change at any instant, with premiums payable continuously.

Define the safe level $\overline{w}(t)$ by

$$\overline{w}(t) = \bar{A}_{x+t}$$

At the safe level the individual can continue to buy insurance for $1 - w$ at each instant, until death.

PROBLEM

Let $\mathcal{R} = \{(w, t) : 0 \leq w \leq \overline{w}(t), \quad 0 \leq t \leq \sup\{\text{ range}(T)\}$

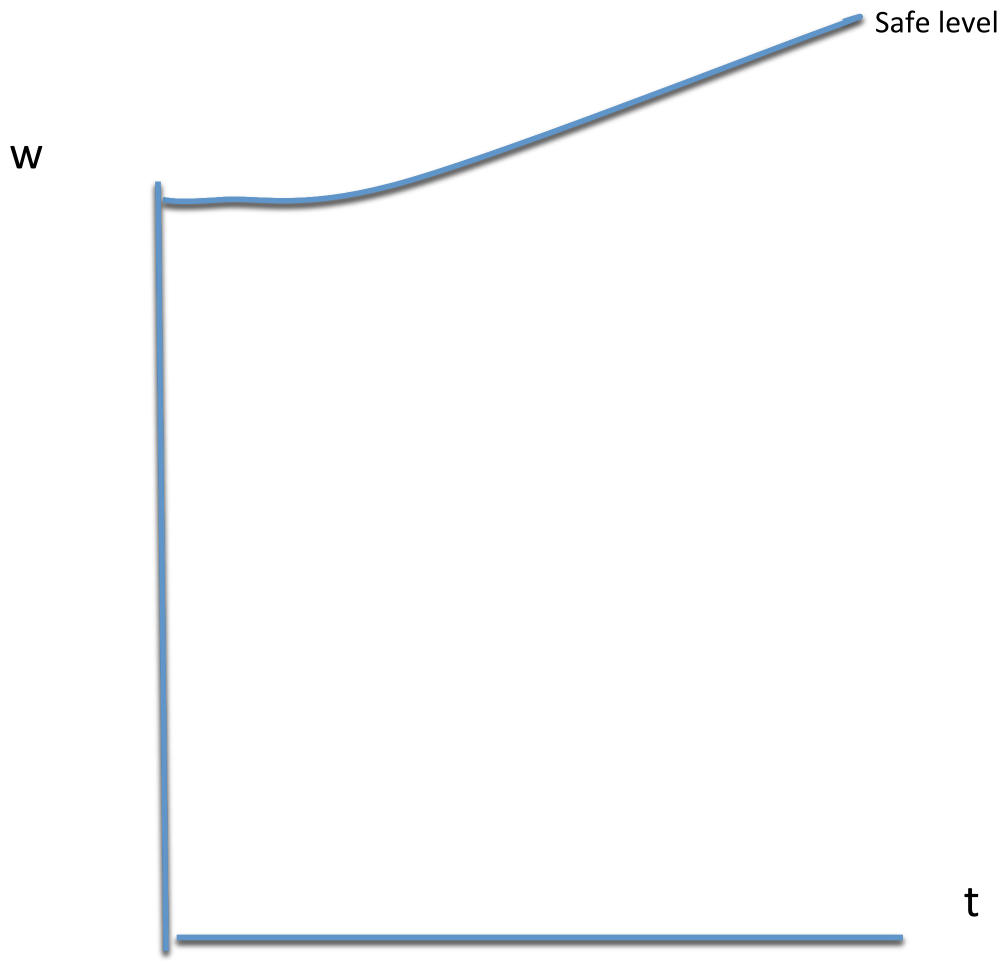
Given any point $(w, t) \in \mathcal{R}$ let

$\phi(w, t) =$ probability of having total wealth of 1 at death, given you have wealth of w at time t.

We are going to carry insurance of $b(w, t)$ at time t, paying at the instantaneous rate of $b(w, t)(1+\Theta)\lambda(t)$.

We want to choose $b(w, t)$ to maximize $\phi(w, t)$.

It is clear that the optimal $b(w, t)$ is either $(1-w)$ or 0, so so the problem reduces to subdividing $\mathcal{R}$ into the buy-subregion and wait-subregion .


Safe level
w
t

DISCRETE-TIME PROBLEM .

Buy insurance for one period at a time, with death benefit payable at end of period. Goal is to maximize probability of having wealth of 1 at the end of the period of death.

Given:

i the rate of interest per period.

q_k = probability of dying between time k and time $k+1$.

$\tilde{q}_k = (1+\Theta)q_k$ (used to calculate premiums).

$N = \max T$.

An individual having wealth of w at time k will either not purchase insurance for the next period, or will purchase insurance with a death benefit of

$$b_k = \frac{1-(1+i)w}{1-\tilde{q}_k}$$

Optimal ϕ is found by dynamic programming .

$$\phi(w, N-1) = \begin{cases} 1 & \text{if } w \geq (1+i)^{-1} \\ 0 & \text{if } w < (1+i)^{-1} \end{cases}$$

$\phi(w, k) =$ maximum of

$$(1 - q_k)\phi\big(w(1+i), k+1\big)$$

$$q_k + (1 - q_k)\phi\left(\frac{w(1+i) - \tilde{q}_{k+1}}{1 - \tilde{q}_{k+1}}, k+1\right)$$

The optimal strategy is not to purchase at time k when the maximum is the firm or to purchase when the maximum is the term on the right.

CONTROL EQUATION:

$$\lambda(t)\phi = \phi_t + rw\phi_w + \max[\lambda(t) - (1+\Theta)\lambda(t)(1-w)\phi_w, 0],$$

with boundary values $\phi(0,t) = 0, \phi(\overline{w}(t), t) = 1.$

POSSIBLE SOLUTIONS

A. Define $\rho(w,t)$ by

$$w = A^{1}_{x+t:\overline{\rho(w,t)|}}$$

and let

$$\phi^{f}(w,t) = 1 - \frac{s\big(\rho(w,t)\big)}{s(t)}$$

Corresponds to strategy of continuing to buy insurance until death or ruin.

Satisfies control equation with first term as max.

B. For any one variable function τ define

$$ {}^{\tau}\phi(w,t) = \frac{s\big(\tau(we^{-rt})\big)}{s(t)} \phi^{f}\big(we^{r(\tau(we^{-rt}-t)}, \tau(we^{-rt})\big). $$

This corresponds to strategy of waiting until time $\tau(w_0)$,where w_0 is the wealth at time 0 which would accumulate to w at time t, and then continuing to buy insurance until death or ruin. ${}^{\tau}\phi$ satisfies the control equation with the second term under the max.

In particular if τ satisfies

$$ we^{r\tau(w)} = \overline{w}(\tau(w)) $$

this means waiting until the safe level.

MAIN CONCLUSIONS:

Given (w, t)

1. If $r \geq \lambda(u)$ for all $u \geq t$, wait .

2. If $\Theta = 0$ and $r \leq g(u)/s(t)$ for all $u \geq t$, buy.

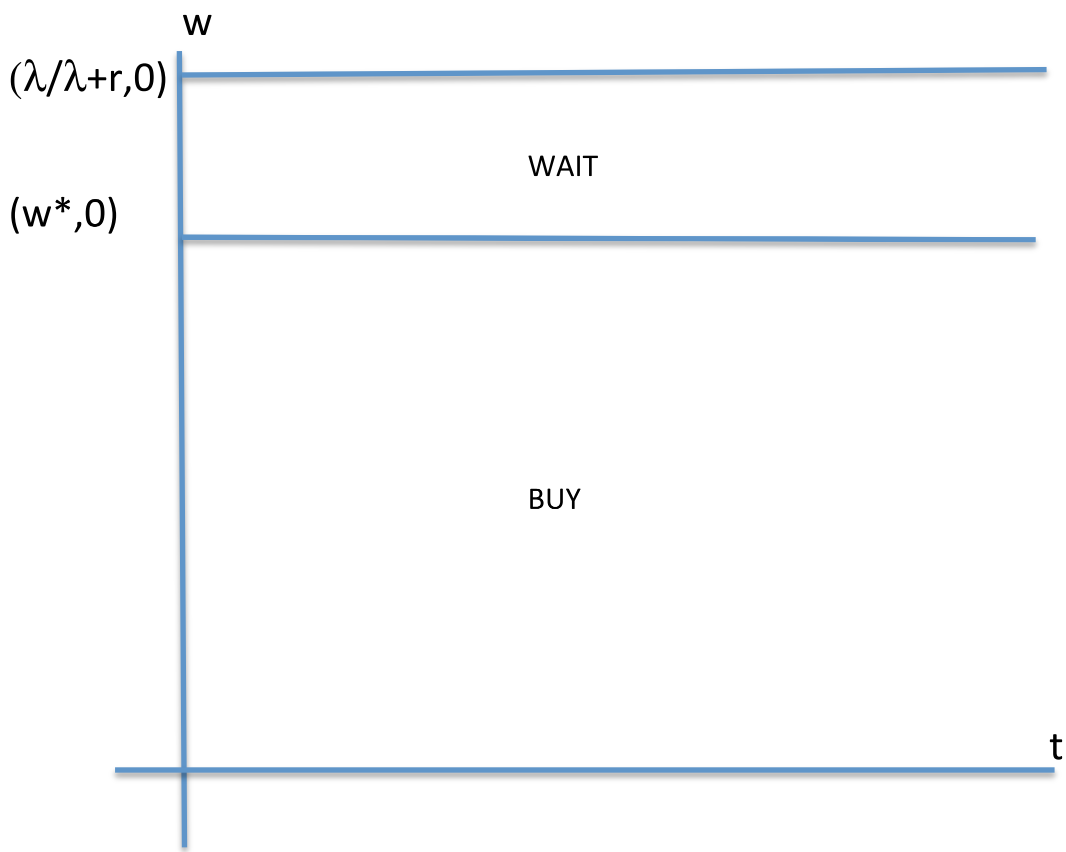
3. If $\lambda(t)$ is a constant $\lambda \geq r$ there exists a constant $w^* < 1$ such that:

Buy if $w < w^*$,

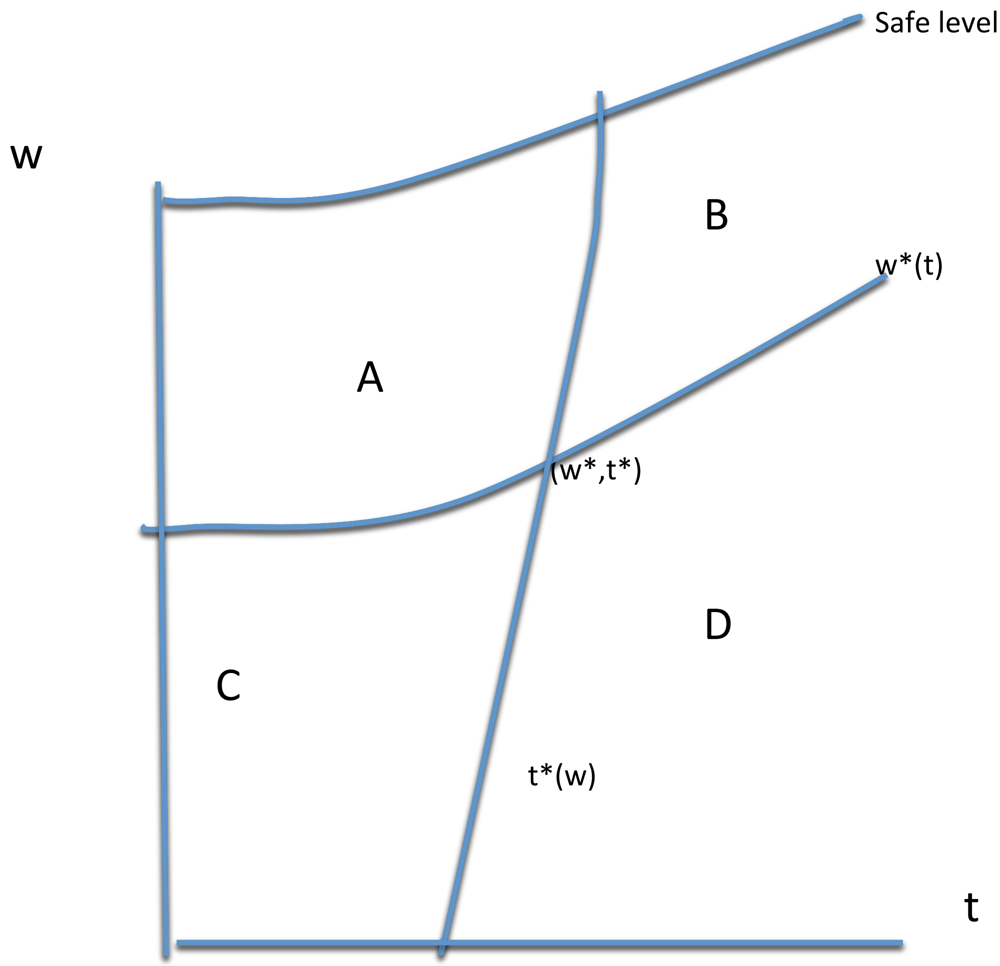
Wait if $w > w^*$

4. $\Theta = 0$, g is increasing ,

wait if $t < t_r$, buy if $t > t_r$
where $r = \lambda(t_r)$.



Constant hazard greater than the force of interest



A,B: Wait until safe level

C: Wait until $t^*(we^{-rt})$

D: Buy

For $s < t^*$, $w^*(s) = w^* e^{-r(t^*-s)}$

OTHER CRITERIA

Maximize the expected value of

$$\min(\text{ wealth at death }, 1)$$

Optimal strategy is always to wait until safe level.