
Presenter: Sharon S. Yang
National Central University, Taiwan
Outline

- Introduction and Literature Review
- The Valuation Framework for HECM Program
  - Non-recourse provision
  - Mortgage insurance
- Modeling Joint-Life Mortality
  - Copula model
- Numerical Analysis
- Conclusion
Introduction
What are Reverse Mortgage Products?

- A kind of home equity conversion that allows the elder persons to borrow money with their home as the collateral.
- The loans accrue interest are only repaid once the people is died or leave the house. → No Fixed Maturity Date
- Loan value is determined by borrower’s age, property value and interest rate
- For example: a rolled-up mortgage (Lump-Sum)

\[
\text{Loan Value: } K \quad \rightarrow \quad K_t = Ke^{rt} \quad \text{at time } t
\]

\[
\text{Property Value: } H_0 \quad \rightarrow \quad H_t
\]
The Risk from Lender Prospective

Key risk

- **Longevity risk**
  - If a borrower lives a longer time than the expected lifespan that may lead the loan balance above the sale proceed of the property.

- **Property risk**
  - If the house price grows at a lower rate than expected, the loan balance may exceed the home value. Lenders may suffer from the losses.

- **Interest rate risk**
  - The rise of interest rates increases the possibility of non-repayment when the loan eventually terminates.

\[ Ke^{v \tau} > H_{\tau} \]
The RM Market: in the U.S.

- Home Equity Conversion Mortgage (HECM) program first introduced in 1989.
- After financial crisis, the private reverse mortgage market has evaporated so that HECM loans represent nearly 100% of newly originated reverse mortgages. (Shen, REE 2011)
- The HECM loan is a non-recourse debt.

Some factors drive the need for RM
- Globally Increasing Life Expectancies
- Many elders are considered to be “Cash Poor & Equity Rich”.
HECM loan

Number of Loans (left axis)  
Property Value (right axis)  
Initial Principal Limit (right axis)
Most of the existing literatures of RMs focus on the following issues:

- **Single-life RMs**
  - Except for Chia and Tsui (2004) work on joint-life RMs
  - Assume mortality is Independent.

- **Lump-Sum payment**
  - Except for Lee et al. (2013) work on tenure RMs

- **Underlying House Price Dynamic**
  - Szymamoski (1994) uses the GBM model.
  - Li et al. (2010) propose the ARMA-EGARCH model.
  - Chen et al. (2010) propose ARMA-GARCH model.
Motivation of this research

- Joint life RMs are getting more and more popular in HECM loans.
- Shemyakin and Youn (1999) point out three possible sources of association between husbands' and wives' mortality.
  * common lifestyle
  * common disaster
  * broken-heart factor
Purpose of this research

- To capture the mortality dependence for pricing joint-life RMIs
  - Lump Sum vs. Tenure

- Pricing Non-Recourse Provisions (NRP) and Mortgage Insurance Premiums (MIP)

- Investigate the effect of mortality dependence on NRP and MIP
Pricing Framework for Joint-life Revere Mortgages
The cash flow of the borrower can be written as follows:

\[
\text{repayment} = \begin{cases} 
-H_t & H_t < L_t \\
-L_t & H_t \geq L_t 
\end{cases} = -L_t + \max(L_t - H_t, 0)
\]

where \( L_t \) is the outstanding balance of the loan and \( H_t \) is the value of the mortgaged property at a time \( t \).

Claim loss function at time \( t \):

\[
CL_t = \max(L_t - H_t, 0)
\]
Payoff of the non-recourse provision

The fair value of the non-recourse provision (NRP) written on a cohort group aged \( x \) can be expressed as the present value of total expected claim losses on a cohort group aged \( x \) as follows:

\[
PV\ ECL = \sum_{t=0}^{\omega-x-1} q_{t|x|} E_Q [e^{-rt} \max(L_t - H_t, 0)]
\]

where

- \( r \) is the risk-free interest rate
- \( t\|q_{x|y} \) is the probability that both a male aged \( x \) and a female aged \( y \) will survive another \( t \) year, but die before next year.
- \( E_Q \) is the expectation under the risk-adjusted measure \( Q \).
Outstanding Balance

- At time $t$, the house value and the accumulated loan balance are
  \[ H_t = H_0 e^{Y_t} \]
  \[ L_t = 0.02H_0 \cdot e^{u_t} + \sum_{j=1}^{t} e^{u(t-j)} 0.005L_j + \sum_{j=0}^{t} A \cdot e^{u(t-j)} \]

- where $A$ is annuity value (unknown).
Mortgage insurance premium

- The present value of mortgage insurance premium of reverse mortgage can be calculated as

\[
PV \text{ MIP} = 0.02H_0 + \sum_{t=1}^{\infty} t p_{xy} e^{-rt}(0.005L_t)
\]

- Initial premium: 2% of the property value
- Continuous premium: 0.5% of outstanding balance

- What does the ratio of PVMIP/NRP mean?
Modeling Joint-Life Mortality
Independent Joint-life Mortality

- We denote the marginal survival functions by $S^m_x$ and $S^f_y$, so far all $t \geq 0$

$$t p^m_x = S^m_x = \Pr[T^m(x) > t]$$
$$t p^f_y = S^f_y = \Pr[T^f(y) > t]$$

Joint survival probability notation

$$t p_{xy} = S_{xy}(t,t) = \Pr[T^m(x) > t \text{ and } T^f(y) > t] \quad \text{indep } S^m_x(t)S^f_y(t)$$

$$t p_{xy} = 1 - t q^m_x \cdot q^f_y = t p^m_x + t p^f_y - t p_{xy}$$

$$t|1 q_{xy} = t p_{xy} - t+1 p_{xy}$$
Correlated Joint-life Mortality

- Shemyakin and Youn (1999) utilized copula functions to calculate joint survival probability for pricing joint life insurance and list three possible sources of association between husbands' and wives' mortality.
  - *common lifestyle*
  - *common disaster*
  - *broken-heart factor*
Correlated Joint-life Mortality

- The joint survival probability can be written as
  \[ t \, p_{xy} = \Pr[T^m(x) > t \text{ and } T^f(y) > t] \dep C(S^m_x(t), S^f_y(t); \theta) \]
  where \( \theta \) is correlated parameter.

- The joint survival probability which the couple are survival exceeding one year can be written as
  \[ p_{x+t,y+t} = S_{x+t}(y+t)(1,1) = C(p_{x+t}, p_{y+t}; \theta_t) \]

- Combing above equation
  \[ t \, p_{xy} = p_{xy} \cdot p_{x+1:y+1} \cdots p_{x+t-1:y+t-1} = \prod_{i=0}^{t-1} C(p_{x+i}, p_{y+i}; \theta_t) \]
Copula Functions

- Gaussian copula (Normal copula):
  \[ C(u_1, u_2; \theta) = \Phi_G(\Phi^{-1}(u_1), \Phi^{-1}(u_2); \theta) \]

- Student-t copula:
  \[ C(u_1, u_2; \theta_1, \theta_2) = t^{\theta_1,\theta_2}_{\theta_1,\theta_2} \{ t^{-1}_{\theta_1}(u_1), t^{-1}_{\theta_1}(u_2) \} \]

- Clayton copula:
  \[ C(u_1, u_2; \theta) = (u_1^{-\theta} + u_2^{-\theta} - 1)^{-\frac{1}{\theta}} \]

- Frank copula:
  \[ C(u_1, u_2; \theta) = -\frac{1}{\theta} \ln[1 + \frac{(e^{-\theta u_1} - 1)(e^{-\theta u_2} - 1)}{e^{-\theta} - 1}] \]

- Gumbel copula:
  \[ C(u_1, u_2; \theta) = \exp[-\{(\log u_1) + (\log u_2)\}]^{-\frac{1}{\theta}} \]
Copula Functions

Gaussian Copula ($\rho = 0.5$)

Student-t Copula ($\rho = 0.5, \nu = 2$)

Clayton Copula ($\alpha = 2$)

Gumbel Copula ($\alpha = 2$)

Frank Copula ($\alpha = 4$)
Lee-Carter Model with Cohort Effect

- Renshaw and Haberman (2006) made cohort-based extension to the LC model.
- \( \ln(m_{xtc}) = \alpha_x + \beta_x(t)\kappa_t + \beta_x(c)\kappa^*_c + \varepsilon_{xt} \)
  - \( m_{xt} \): central death rate for a person aged \( x \) at time \( t \)
  - \( \alpha_x \): describes the average age-specific mortality
  - \( \beta_x \): decline in mortality at age \( x \)
  - \( \kappa_t \): represents the general mortality level
  - \( \kappa^*_c \): cohort effect
  - \( \varepsilon_{x,t} \): deviation of the model from the observed log-central
deadth rates
Modeling the House Price Dynamics

Specifically, the ARMA-GARCH model can be denoted as a ARMA($P$, $Q$) series

$$y_t = c + \sum_{p=1}^{P} b_p y_{t-p} + \sum_{q=1}^{Q} a_q \varepsilon_{t-q} + \varepsilon_t$$

where $P$ is the order of the autocorrelation terms, $Q$ is the order of the moving average terms, $b_p$ is the $p$th-order autocorrelation coefficient, $a_q$ is the $q$th-order moving average coefficient.

Furthermore, white noise, $\varepsilon_{t,j}$, is assumed Gaussian distribution with mean 0 and variance denoted by the GARCH($R$, $M$) model. The model as follows

$$\sigma_{t,j}^2 = \delta_{0,j} + \sum_{r=1}^{R} \delta_{r,j} \varepsilon_{t-r,j}^2 + \sum_{m=1}^{M} \beta_{m,j} \sigma_{t-m,j}^2$$

where $R$ is the order of the GARCH terms, $M$ is the order of the ARCH term, $r$ is the $r$th-order GARCH coefficient, $m$ is the $m$th-order ARCH coefficient, $\delta_{r,j} > 0$ for $r = 1, \ldots, R$ and $\beta_{m,j} > 0$ for $m = 1, \ldots, M$. 
Risk-adjust Probability

- House Price Dynamic
  - Esscher transform (Bühlmann et al. (1996))
  - Lee et al. (2010), Li et al. (2010), Chen et al. (2010)

- Mortality Dynamic
  - Wang Transform
    
    To change the probability measure from the real-world to a risk-neutral measure, Wang (2000) proposes a distortion operator:

    \[ F_n^\tau (x) = \Phi \left( \Phi^{-1} (F_n (x)) + \tau \right) \],

    (18)

  - Lin and Cox (2005), Lee et al. (2010), Yang et al. (2013)
\[ \xi_T = \prod_{t=\Delta t}^{T} \frac{\exp (\varphi (t) \ Y (t))}{E_P (\exp (\varphi (t) \ Y (t)) | F_{t-\Delta t})}. \]

Then, we define a new martingale measure \( Q_\varphi \) by

\[ \frac{dQ_\varphi}{dP} \bigg|_{F_T} = \xi_T. \]

Under the risk-neutral measure \( Q_\varphi \), the housing return becomes

\[ Y (t) = \ln \left( \frac{H (t)}{H (t - \Delta t)} \right) \]

\[ = r (t - \Delta t) \Delta t - \frac{1}{2} h (t) + \varepsilon_H^Q (t), \]
Empirical Study
Fitting House price dynamics

- The house price data
  - Quarterly House Price Index (HPI)
  - 1991Q1~2011Q3
  → ARMA(2,2)-GARCH (1, 1)
Mortality Dynamics

- The mortality rate data
  * U.S., Male and Female

- Data range
  * Age: 60~99
  * Year: 1950~2007

- Source
  * Human Mortality Database (HMD)
  * http://www.mortality.org/
Fitting model (Male)

Figure: Lee-Carter model with cohort estimated parameters
Fitting model (Female)

Figure: Lee-Carter model with cohort estimated parameters
Selection of Copula Models

- Comparing with AIC and BIC, the Clayton and Gumbel copulas are selected for difference ages.
  - Clayton copula is selected for 60~72 age and Gumbel copula is selected for 73~95 age.
- Trivedi and Zimmer (2005) suggested, when correlation between spouses' age at death is strongest in the left tail of the joint distribution, Clayton is an appropriate modeling choice.
Joint-life Probability at same age (xx)
Assumptions for Calculating NRP and MIP

- Assume house price independent mortality rate.
- Male and Female are at the same age.
- $H_0 = 300,000$
- $r = 1.7523\%.$
- $u = 2.451\%.$
- $g = 2\%.$
- $K = 6\%.$
- $\omega = 99.$

- We use Monte Carlo simulation 10,000 times and utilize Antithetic variance reduction to reduce the variation of pricing the reverse mortgage.
**NRP and MIP**: lump sum and tenure payment

Note: the \( \cdots \) lines and \( \cdots \) lines are the present values of mortgage insurance premiums (MIP) and the value of the non-recourse provision (NRP), respectively.

Figure 7: Values of the NRP and MIP under products of lump sum (right figure) and tenure payment (left figure).
MIP/NRP: lump sum and tenure payment

Note: the — o — lines and — o — lines are the present values of mortgage insurance premiums (MIP) and the value of the non-recourse provision (NRP), respectively.

Figure 8: Ratios of MIP to NRP under products of lump sum (right figure) and tenure payment (left figure).
Initial house price value

Figure 10: Values of the NRP and MIP under independent situation (right figure) and copula function (left figure) in 2010.
Initial house price value

Figure 12: Ratios of MIP to NRP under independent situation (right figure) and copula function (left figure) in 2010.
Conclusion

- Ignoring the dependence between joint-life mortality overestimates the NRP and MIP
  - Such effect is more significant for lump sum RM

- The insurance company can use their own mortality experience to replace the empirical analysis.
  - Frees et al. (1996)
  - Luciano, et al. (2008) model the mortality risk of couples of individuals, according to the stochastic intensity approach.
Thank you!

Q&A