Valuing Variable Annuity Guarantees on Multiple Assets

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Outline of the Presentation

- Basics
- Motivation
- The GMMB and GMDB riders
- Pricing through Fourier transforms
- Numerical implementation
- Numerical results

Basics

- Variable annuities fulfill the social needs for the aging population by providing products that deliver certainty of income upon retirement.
- Unlike traditional mutual funds and life insurance products, variable annuity contracts come with embedded guarantees which protect the policyholder's savings against unanticipated outcomes.
- Guarantees can be underwritten for the accumulation phase, annuity phase or untimely death of the policyholder, and they fall into two major groups i.e. GMDB and GMLB.
- GMLB can further be categorized into GMxB where where x stands for maturity (M), income (I) and withdrawal (W).
- Most of the research has focused on guarantees structured on a single underlying asset whose dynamics follow the standard geometric Brownian motion proposed in Black and Scholes (1973).

Motivation

- Milevsky and Posner (2001) derive semi-analytical expressions for the valuation of GMDB riders using risk-neutral techniques.
- Bacinello et al. (2012) devise a general framework for valuing various types of guarantees using ordinary and least squares Monte Carlo methods
- Contrary to the single underlying asset feature, Ng and Li (2011) note that in practice most variable annuity guarantees are written on multiple sub-account funds, and the correlations between funds can be material.
- They propose a multivariate regime-switching framework for modelling the joint returns on various assets and use Monte-Carlo based algorithms to price GMMB and GMDB riders when the underlying fund is made up of the two assets.
- We develop an analytical framework for valuing GMMB and GMDB riders structured on several underlying funds whose dynamics evolve according to stochastic volatility processes of the affine type proposed in Heston (1993).

The Model-The Financial Assets

- We consider a fund that involves a choice between two assets such that

$$F(T) = H(T, s_1(T), s_2(T))(1-m)^T. \tag{1}$$

- Here, $H(T, s_1(T), s_2(T))$ is any payoff function and the risk neutral dynamics of the two assets

$$ds_1 = rs_1 dt + \sqrt{v_0} s_1 dw_0 + \sqrt{v_1} s_1 dw_1, \tag{2}$$
$$ds_2 = rs_2 dt + \sqrt{v_0} s_2 dw_0 + \sqrt{v_2} s_2 dw_2, \tag{3}$$

where

$$dv_0 = \kappa_0(\theta_0 - v_0)dt + \sigma_0\sqrt{v_0}dz_0, \tag{4}$$
$$dv_1 = \kappa_1(\theta_1 - v_1)dt + \sigma_1\sqrt{v_1}dz_1, \tag{5}$$
$$dv_2 = \kappa_2(\theta_2 - v_2)dt + \sigma_2\sqrt{v_2}dz_2, \tag{6}$$

- We assume that $dw_0 dz_0 = \rho_0 dt$, $dw_1 dz_1 = \rho_1 dt$ and $dw_2 dz_2 = \rho_2 dt$ and all other correlations are assumed to be equal to zero.

- the correlation between the two assets is given by

$$\boxed{d\,\mathrm{Corr}(\ln s_1, \ln s_2)_t = \frac{v_0}{\sqrt{v_0 + v_1}\sqrt{v_0 + v_2}} dt, \qquad (7)}$$

  which leads to a mean long term correlation around the value $\frac{\theta_0}{\sqrt{\theta_0+\theta_1}\sqrt{\theta_0+\theta_2}}$. If we restrict the model to a single asset, that is to say to equations (2), (4) and (5) then this model is similar to the one proposed in Christoffersen et al. (2009) and its earlier version with jumps presented in Bates (2000).

- Characteristic function known in closed form, by letting $(x_1(\tau), x_2(\tau)) = (\ln(s_1(\tau)), \ln(s_2(\tau)))$ we obtain

$$\mathbb{E}^{Q}[e^{iz_1x_1(\tau)+ix_2(\tau)}] \\ = e^{iz_1x_1(\tau)+iz_2x_2(\tau)+iz_1r\tau+iz_2r\tau+a(\tau)+b_0(\tau)v_0(\tau)+b_1(\tau)v_1(\tau)+b_2(\tau)v_2(\tau)}, \tag{8}$$

with $\tau = T - t$ and

$$a(\tau) = a_0(\tau) + a_1(\tau) + a_2(\tau), \tag{9}$$

$$a_j(\tau) = \frac{2\kappa_j\theta_j}{\sigma_j^2}\left(\tau\lambda_-^j - \log\left(\frac{\lambda_+^j - \lambda_-^j e^{-\sqrt{\Delta_j}\tau}}{\lambda_+^j - \lambda_-^j}\right)\right) \quad j = 0, 1, 2, \tag{10}$$

$$b_j(\tau) = -\eta_j \frac{1 - e^{-\sqrt{\Delta_j}\tau}}{\lambda_+^j - \lambda_-^j e^{-\sqrt{\Delta_j}\tau}} \quad j = 0, 1, 2, \tag{11}$$

where a_j and b_j for j=1,2 are algebraic funtions.

The Model-The Mortality Process

- We adopt the time-inhomogeneous affine mortality process as presented in Ziveyi et al. (2013) such that

$$d\mu(t;x) = \kappa_\mu(m(t) - \mu(t;x))dt + \sigma_\mu\sqrt{\mu(t;x)}dW(t), \tag{12}$$

where

$$\sigma_\mu = \Sigma_\mu\sqrt{m(t)}.$$

- Biffis (2005) chooses $m(t)$ to be a deterministic function given by

$$m(x+t) = \frac{c}{\theta^c}(x+t)^{c-1}, \tag{13}$$

which is the Weibull mortality law.

- The corresponding survival probability can be shown to be

$${}_{T-t}p_{x+t} = e^{\alpha_\mu(t,T;x)-\beta_\mu(t,T;x)\mu(x,t)}, \tag{14}$$

where $\alpha_\mu(t,T;x)$ and $\beta_\mu(t,T;x)$ are solutions of respective characteristic PDEs.

The Variable Annuity Guarantees

- We now value GMMB and GMDB riders embedded in variable annuities.
- The GMDB is a natural extension of the GMMB as will be shown below.
- Denoting the fund value at initial time as $F(0)$ and the guarantee rate as g, the minimum payout at maturity of the contract can be represented as $F(0)e^{gT}$.
- The value of a GMMB rider can be represented as

$$
\begin{aligned}
V_M(x,t,T) &= \mathbb{E}_t^Q\left[\mathbb{1}_{\{T_x > T\}} e^{-\int_t^T r(s)ds} H(T)\right] \\
&= \mathbb{E}_t^Q\left[e^{-\int_t^T [r(s)+\mu(s)]ds} H(T)\right] \\
&= {}_{T-t}p_{x+t}\mathbb{E}_t^Q\left[e^{-\int_t^T r(s)ds} H(T)\right] \\
&= {}_{T-t}p_{x+t}B(t,T)\mathbb{E}_t^Q\left[H(T)\right] \\
&= {}_{T-t}p_{x+t}V(t,T), \qquad (15)
\end{aligned}
$$

Valuing GMMB on the weighted sum of two assets

- Consider the initial fund value of

$$\boxed{F(0) = \omega_1 s_1(0) + \omega_2 s_2(0), \qquad (16)}$$

- The payoff of the GMMB at maturity time T can then be represented as

$$\boxed{H(T) = (K - (\omega_1 s_1(T) + \omega_2 s_2(T)))^+ \qquad (17)}$$

where $K = F(0)e^{gT}$ with g being the guarantee rate.

- From equation (15), the value of a GMMB involves the computation of

$$\boxed{V(0,T) = B(0,T)\mathbb{E}^Q\left[(K - (\omega_1 s_1(T) + \omega_2 s_2(T)))^+\right]. \qquad (18)}$$

Valuing GMMB on the best of two assets

- The payoff of the GMMB at maturity time T can then be represented as
$$\boxed{H(T) = \left(F(0)e^{gT} - \max(s_1(T), s_2(T))\right)^+, \qquad (19)}$$
where as before $K = F(0)e^{gT}$ with g being the guarantee rate.
- Without loss of generality we suppose that $F(0) = \max(s_1(0), s_2(0))$. Equation (15) then implies that the corresponding value of the GMMB can be shown to be
$$\boxed{V(0,T) = B(0,T)\mathbb{E}^Q\left[(K - \max(s_1(T), s_2(T)))^+\right]. \quad (20)}$$

Pricing GMMBs through Fourier transforms

- The pricing equation can be rewritten as

$$\begin{aligned} V(0,T) &= B(0,T)\mathbb{E}^{Q}\left[h(\ln s_1(T), \ln s_2(T))\right] \\ &= B(0,T)\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} h(x_1,x_2)f(x_1,x_2)dx_1dx_2 \end{aligned} \tag{21}$$

where $f(x_1,x_2)$ stands for the density of $(\ln s_1(T), \ln s_2(T))$.

- By definition of the characteristic function we have

$$f(x_1,x_2) = \frac{1}{(2\pi)^2}\int_{\mathbb{C}^2} e^{-ix_1z_1-ix_2z_2}\phi(0,T,z_1,z_2)dz_1dz_2.$$

- Inserting this equality in the equation (21) yields

$$V(0,T) = \frac{B(0,T)}{(2\pi)^2}\int_{\mathbb{C}^2}\phi(0,T,z_1,z_2)\hat{h}(z_1,z_2)dz_1dz_2 \tag{22}$$

where

$$\hat{h}(z_1,z_2) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-ix_1z_1-ix_2z_2}h(x_1,x_2)dx_1dx_2. \tag{23}$$

Fourier transforms of the payoff functions

- In the case of the weighted sum of assets if we assume $\omega_1 > 0$ and $\omega_2 > 0$, the Fourier transform of the payoff function can be represented as

$$\boxed{\hat{h}(z_1, z_2) = \frac{\omega_2^{iz_2}\omega_1^{iz_1} K^{1-iz_1-iz_2}}{(iz_2 - 1)(iz_2)} \frac{\Gamma(-iz_1)\Gamma(2 - iz_2)}{\Gamma(2 - iz_1 - iz_2)}, \qquad (24)}$$

with $\Im(z_2) > 0$.

- The Fourier transform of the payoff involving the best of two assets can be represented as

$$\boxed{\begin{aligned}\hat{h}(z_1, z_2) = K^{1-iz_1-iz_2} \Bigg(& \frac{1}{(iz_1 + iz_2 - 1)(iz_2 - 1)} + \frac{1}{(z_1 + z_2)z_2} \\ & + \frac{1}{z_1 z_2 (iz_2 - 1)} + \frac{1}{z_2(z_1 + z_2)(iz_1 + iz_2 - 1)} \Bigg) \\ & \qquad (25)\end{aligned}}$$

with the constraints that $\Im(z_2) > 0$ and $\Im(z_1 + z_2) > 0$.

Pricing GMDBs

- From above computations, the value of the guaranteed minimum death benefit (GMDB) rider can also be obtained
- The value of the death benefit $H(\tau_x)$, payable in case the policyholder dies before time T, can be represented as

$$\boxed{\begin{aligned} V_D(x,0,T) &= \mathbb{E}^Q\left[e^{-\int_0^{\tau_x} r(s)ds} H(\tau_x)\mathbf{1}_{\{t\le\tau_x\le T\}}\right] \\ &= \mathbf{1}_{\{\tau_x>0\}}\int_0^T \mathbb{E}^Q\left[e^{-\int_0^u r(s)+\mu(s)ds}\mu(u)H(u)\right]du \\ &= \mathbf{1}_{\{\tau_x>0\}}\int_0^T \mathbb{E}^Q\left[e^{-\int_0^u \mu(s)ds}\mu(u)\right]\mathbb{E}^Q\left[e^{-\int_0^u r(s)ds}H(u)\right]du \\ &= \mathbf{1}_{\{\tau_x>0\}}\int_0^T \mathbb{E}^Q\left[e^{-\int_0^u \mu(s)ds}\mu(u)\right]V(0,u)du, \qquad (26) \end{aligned}}$$

where $0\le\tau_x\le T$ and $H(u)$ is the payoff function as presented in equation (17) for the case of weighted sum of assets or (19) in the case of the best performing asset.

Computing the value of the GMDB

- To compute the expectation in (26) we define the function $G(t, T, z) = \mathbb{E}_t^Q \left[e^{z\mu(T) - \int_t^T \mu(s)ds} \right]$ which is similar to the survival functon with $\alpha_\mu(t, T; x)$ and $\beta_\mu(t, T; x)$ being solutions to ODEs.
- Once this function is known then the expectation involved in (26) is given by $\partial_z G(0, u, z)|_{z=0}$.
- We implement the discretized version such that

$$\boxed{\begin{aligned} V_D(x, 0, T) &= \sum_{i=1}^{N} \mathbb{E}^Q \left[\mathbf{1}_{\{\tau_x \in [t_{i-1}\ t_i]\}} e^{-\int_0^{t_i} r(s)ds} H(t_i) \right] \\ &= \sum_{i=1}^{N} \mathbb{E}^Q \left[\mathbf{1}_{\{\tau_x \in [t_{i-1}\ t_i]\}} \right] \mathbb{E}^Q \left[e^{-\int_0^{t_i} r(s)ds} H(t_i) \right] \\ &= \sum_{i=1}^{N} \left({}_{t_{i-1}}p_x - {}_{t_i}p_x \right) V(0, t_i). \qquad (27) \end{aligned}}$$

- We approximate the double integral in equation (22) with a double sum over the lattice

$$\boxed{\begin{aligned}\Gamma &= \left\{z(k) = (z_1(k_1), z_2(k_2)) | k = (k_1, k_2) \in \{0, \cdots, N-1\}^2\right\},\\ z(k) &= -\bar{z} + k\eta. \qquad (28)\end{aligned}}$$

- An approximation of the option price component is then given by

$$\boxed{V(0,T) \approx \frac{\eta^2 B(0,T)}{(2\pi)^2} \sum_{k_1=0}^{N-1} \sum_{k_2=0}^{N-1} \phi(0, T, z(k) + i\epsilon)\hat{h}(z(k) + i\epsilon), \qquad (29)}$$

where $\epsilon \in \mathbb{R}^2$ is a vector such that the Fourier transform of the considered payoff is well defined.

Distribution of log-returns

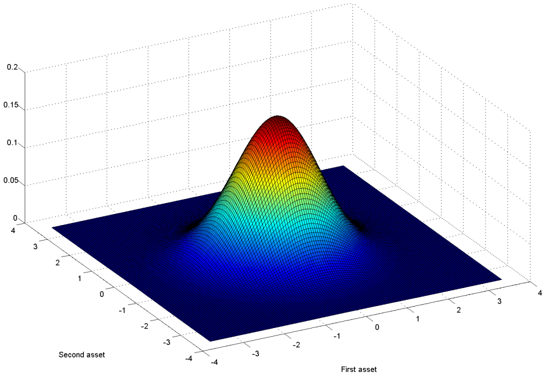

Figure: 5Y log-return distribution of two assets for the "Low correlation" parameter set .

Distribution of log-returns

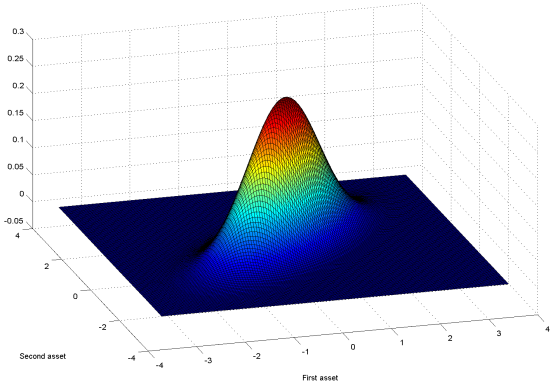

Figure: 5Y log-return distribution of two assets for the "High correlation" parameter set .

GMMB prices for the High & Low correlation parameter sets

g (%)	Low correl. T (in years) 5	Low correl. T (in years) 10	High correl. T (in years) 5	High correl. T (in years) 10
Age at inception: 50				
1	0.14412	0.14659	0.19609	0.19526
2	0.16424	0.17995	0.21848	0.23267
3	0.1866	0.21958	0.24296	0.2762
4	0.21138	0.26638	0.26969	0.32669
5	0.23873	0.32133	0.2988	0.38502
Age at inception: 60				
1	0.14189	0.14139	0.19305	0.18833
2	0.1617	0.17357	0.2151	0.22441
3	0.18372	0.21179	0.2392	0.26641
4	0.20811	0.25693	0.26551	0.3151
5	0.23504	0.30993	0.29418	0.37136

Table: GMMB prices for the weighted-sum payoff ($\omega_1 = \omega_2 = 0.5$).

GMMB prices for the High & Low correlation parameter sets cont...

	Low correl.		High correl.	
g (%)	T (in years)		T (in years)	
	5	10	5	10
Age at inception: 50				
1	0.086033	0.096416	0.15051	0.15677
2	0.099328	0.12032	0.16904	0.18854
3	0.11433	0.14926	0.1895	0.22588
4	0.1312	0.18408	0.21202	0.26959
5	0.15015	0.22573	0.23678	0.32056
Age at inception: 60				
1	0.084702	0.092995	0.14818	0.15121
2	0.097791	0.11606	0.16643	0.18185
3	0.11256	0.14397	0.18657	0.21787
4	0.12919	0.17755	0.20874	0.26002
5	0.14783	0.21772	0.23311	0.30919

Table: GMMB prices for the best-of payoff.

GMDB prices for the High & Low correlation parameter sets

	Low correl.		High correl.	
g (%)	T (in years)		T (in years)	
	5	10	5	10
Age at inception: 50				
1	0.000422	0.00120	0.000568	0.001586
2	0.000497	0.00155	0.000648	0.001966
3	0.000582	0.00198	0.000738	0.002420
4	0.000678	0.00251	0.000837	0.002957
5	0.000786	0.00313	0.000946	0.003585
Age at inception: 60				
1	0.002485	0.00670	0.003342	0.008850
2	0.002928	0.00870	0.003817	0.010995
3	0.003431	0.01116	0.004346	0.013557
4	0.003999	0.01412	0.004932	0.016589
5	0.004636	0.01764	0.005579	0.020139

Table: GMDB prices for the weighted-sum payoff ($\omega_1 = \omega_2 = 0.5$).

GMDB prices for the High & Low correlation parameter sets cont...

g (%)	Low correl. T (in years) 5	Low correl. T (in years) 10	High correl. T (in years) 5	High correl. T (in years) 10
Age at inception: 50				
1	0.000237	0.000735	0.000428	0.001239
2	0.000284	0.000975	0.000493	0.001558
3	0.000339	0.001280	0.000567	0.001945
4	0.000403	0.001664	0.000651	0.002411
5	0.000476	0.002136	0.000744	0.002966
Age at inception: 60				
1	0.001393	0.004089	0.0025208	0.006905
2	0.001671	0.005442	0.0029086	0.008702
3	0.001996	0.007169	0.0033458	0.010889
4	0.002373	0.009339	0.0038365	0.013523
5	0.002807	0.012016	0.0043848	0.016661

Table: GMDB prices for the best-of payoff.

Questions and Comments?