Valuing Guaranteed Minimum Death Benefits in Variable Annuities with Knock-Out Options

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1. The motivation – basic problem
2. Lapses and surrenders incorporation
3. Main results – value up-and-out option
4. Main results – a generalization of our formula

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we are to value a $\boldsymbol{K}$**-strike put option** that is exercised at time T_x.

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- δ denotes a force of interest.

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Exponential case is sufficient

Our problem is reduced to finding

$$\mathsf{E}[e^{-\delta T} b(S(T))],$$

where T is an **exponential** random variable **independent** of the stock price process $\{S(t)\}$.

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$$1(\max_{0 \le t \le T} S(t) < U)[K - S(T)]^+,$$

where $1(.)$ is an indicator function and U is a barrier.

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This is the payoff of an **up-and-out put option**.

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Our valuation problem with lapses and surrenders

We aim to determine the following expected present value for an **up-and-out option**,

$$\mathsf{E}[e^{-\delta T} 1(\max_{0 \le t \le T} S(t) < U) b(S(T))],$$

where $b(.)$ is a death benefit function, and T is an **independent exponential** exercise date with mean $1/\lambda$.

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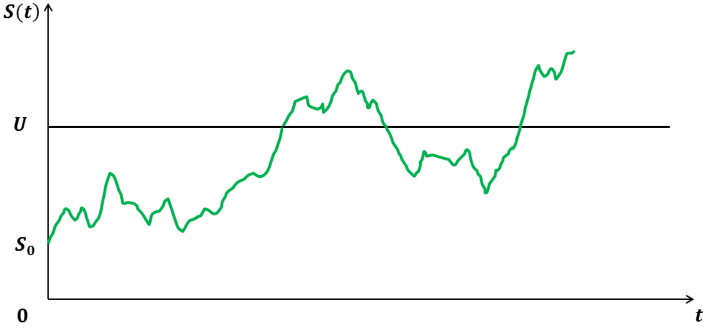

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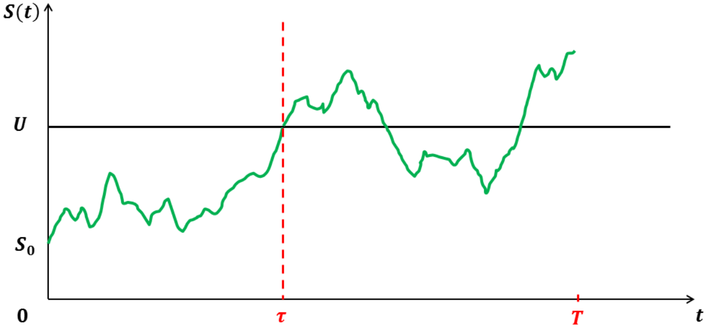

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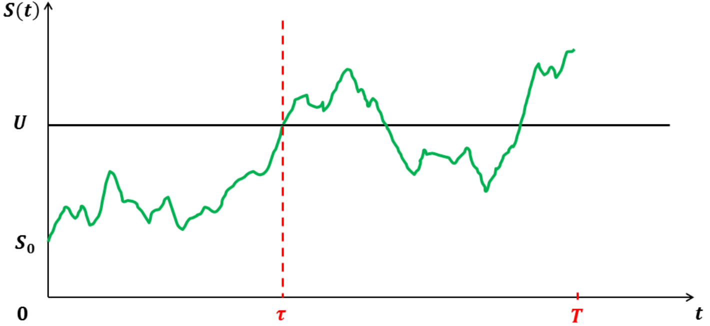

Note that $\max_{0 \leq t \leq T} S(t) < U \Leftrightarrow \tau > T$, or $\max_{0 \leq t \leq T} S(t) \geq U \Leftrightarrow \tau \leq T$.

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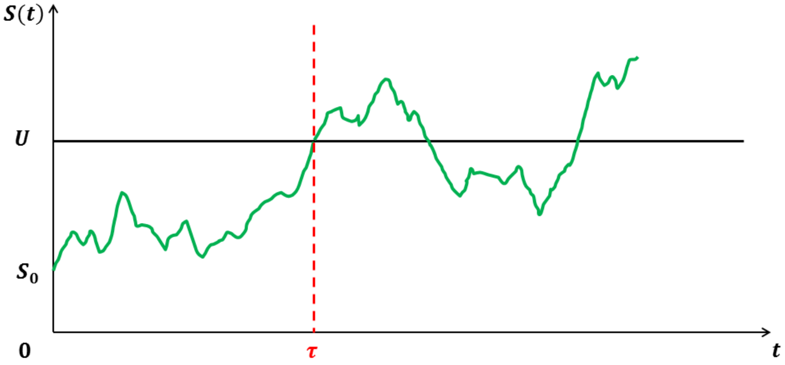

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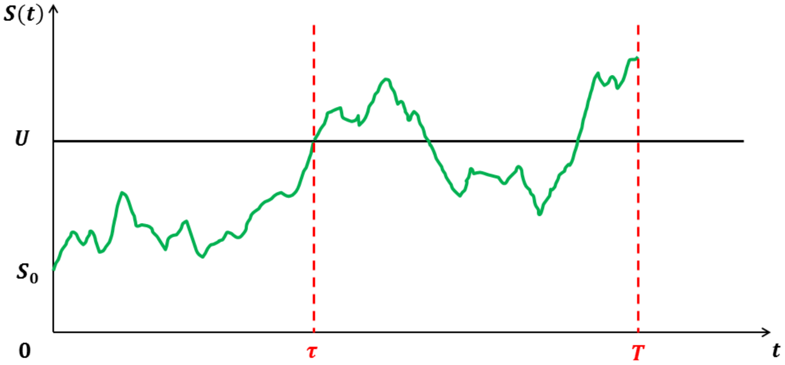

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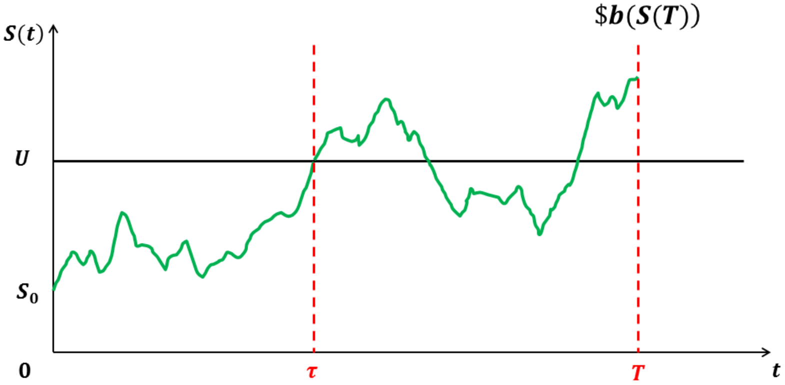

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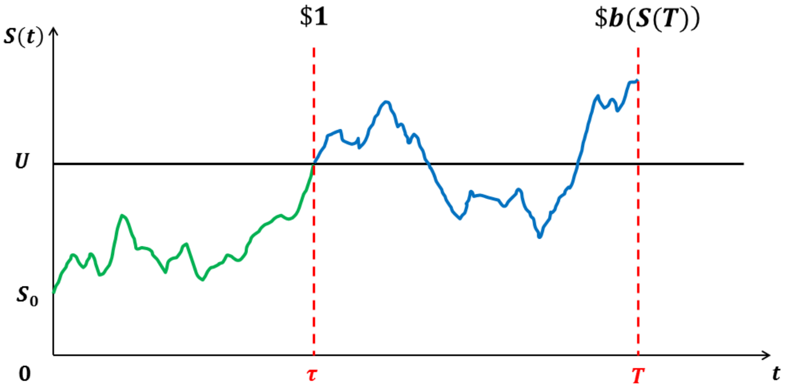

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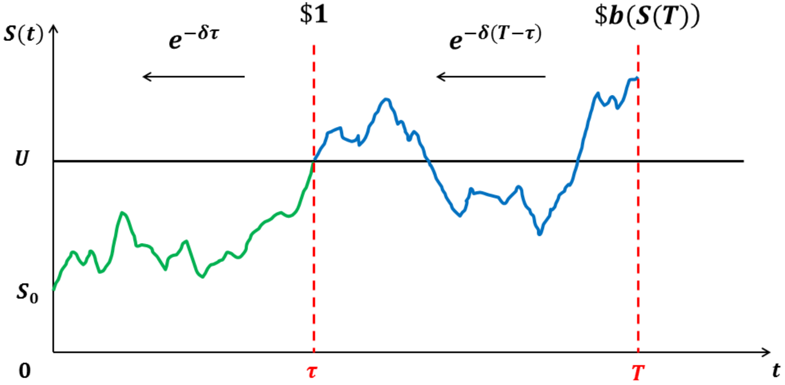

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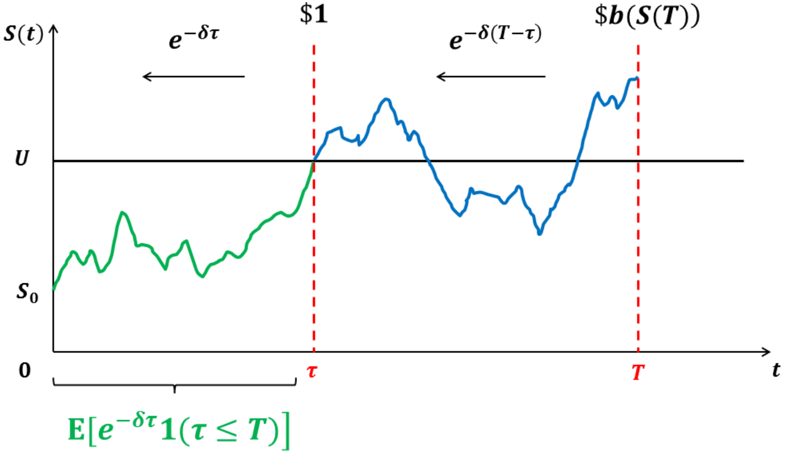

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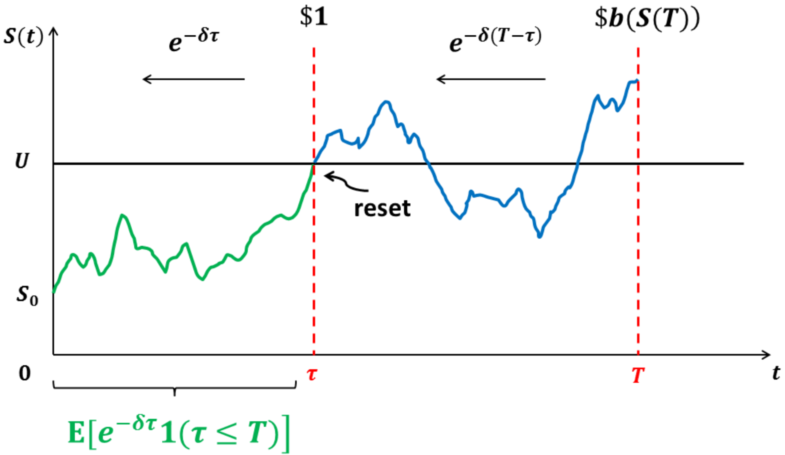

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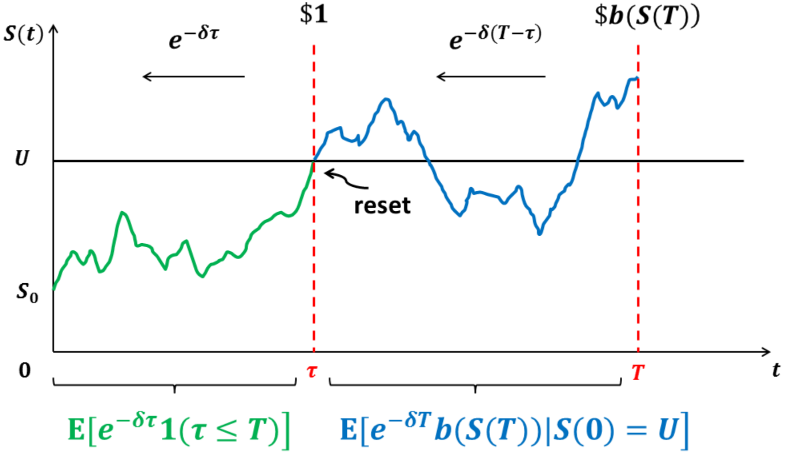

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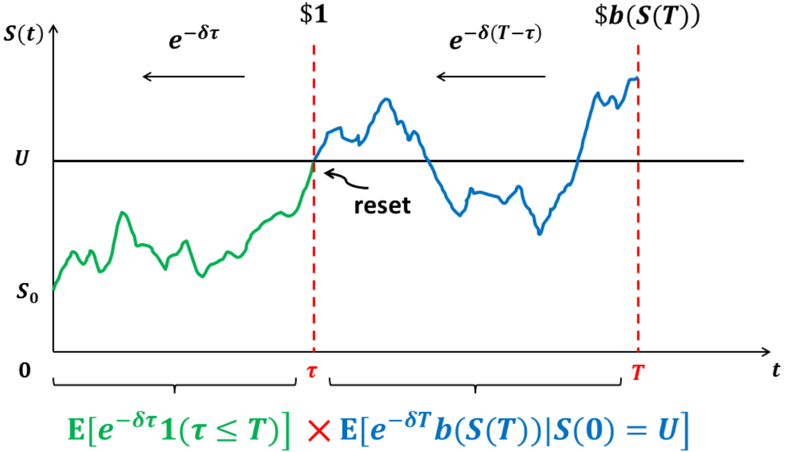

The formula for up-and-in option

$$\mathsf{E}[e^{-\delta T}1(\tau \leq T)b(S(T))] = \mathsf{E}[e^{-\delta\tau}1(\tau \leq T)] \times \mathsf{E}[e^{-\delta T}b(S(T))|S(0) = U]$$

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Formula for up-and-in option

For $S_0 < U$,

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For $S_0 < U$,

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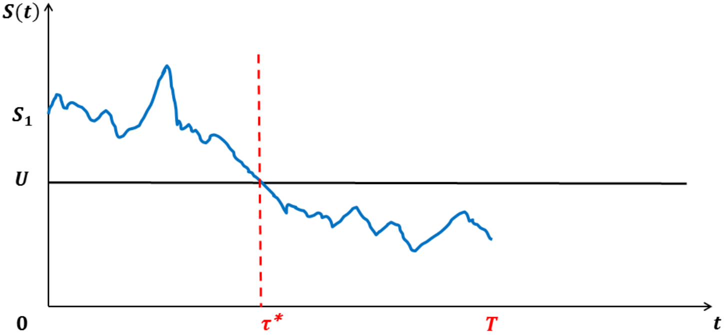

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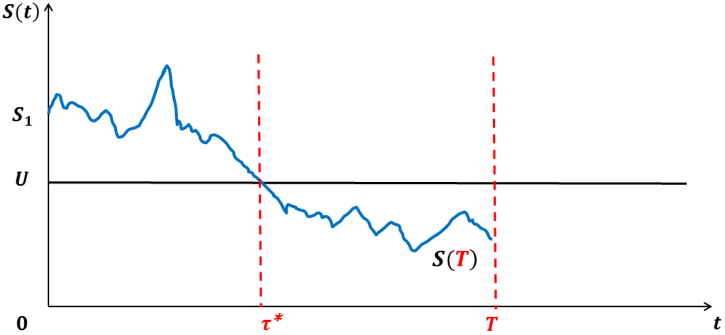

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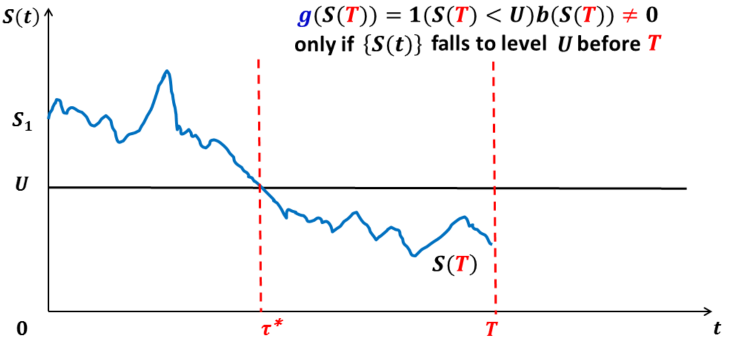

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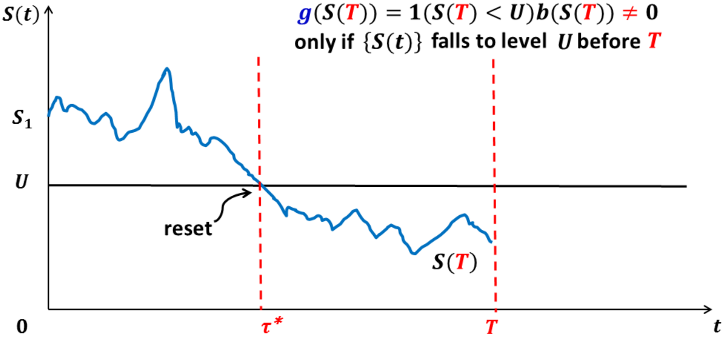

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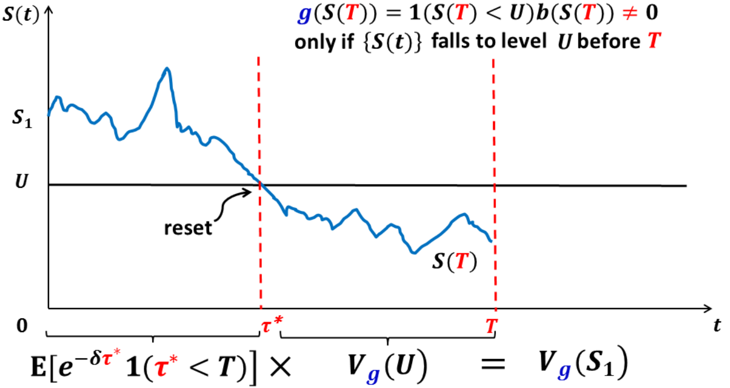

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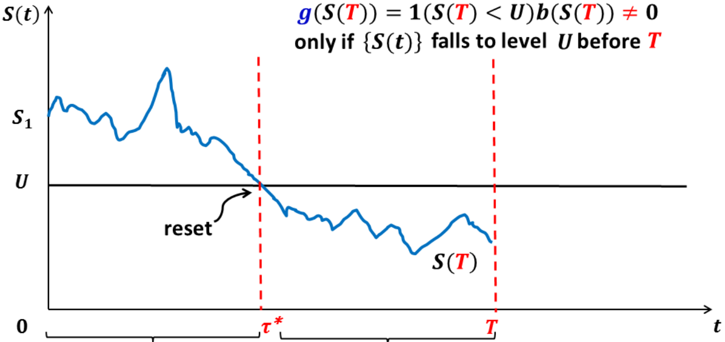

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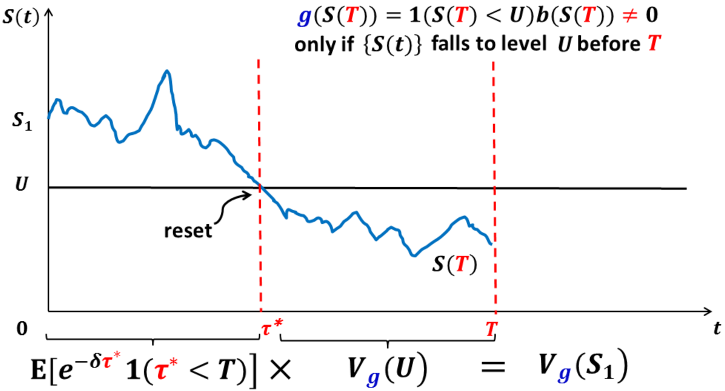

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Generalized formula for up-and-out option

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Hence, a generalization.

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- Single and double barrier options with **exponentially curved boundaries**. Example, $U(t) = Ue^{\theta t}, t \geq 0$.

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Thank you!

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Denote the two roots by $\theta^+ > 0$ and $\theta^- < 0$.

Find E$[e^{-\delta\tau}1(\tau \leq T)]$ cont'd

Recall that τ is the **first** time when $\{S(t)\}$ **rises** to level U.

Find $\mathrm{E}[e^{-\delta\tau} 1(\tau \leq T)]$ cont'd

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Because of **Optional Sampling Theorem**,

Find E$[e^{-\delta\tau}1(\tau \leq T)]$ cont'd

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Because of **Optional Sampling Theorem**, if we replace fixed t by τ, the condition for martingale still holds for θ^+,

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Because of **Optional Sampling Theorem**, if we replace fixed t by τ, the condition for martingale still holds for θ^+,

$$\mathsf{E}[e^{-\delta\tau}1(\tau \le T)[S(\tau)]^{\theta^+}] = [S(0)]^{\theta^+},$$

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Similarly, define τ^{*} as the **first** time when $\{S(t)\}$ **falls** to level U with an initial price $S_1 > U$.

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Find $\mathsf{E}[e^{-\delta\tau}1(\tau \le T)]$ cont'd

Recall that τ is the **first** time when $\{S(t)\}$ **rises** to level U.
Because of **Optional Sampling Theorem**, if we replace fixed t by τ, the condition for martingale still holds for θ^+,

$$\mathsf{E}[e^{-\delta\tau}1(\tau \le T)[S(\tau)]^{\theta^+}] = [S(0)]^{\theta^+},$$

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$$\Leftrightarrow \quad \mathsf{E}[e^{-\delta\tau}1(\tau \le T)] = \left[\frac{S_0}{U}\right]^{\theta^+}.$$

Similarly, define τ^* as the **first** time when $\{S(t)\}$ **falls** to level U with an initial price $S_1 > U$. Then

$$\mathsf{E}[e^{-\delta\tau^*}1(\tau^* \le T)] = \left[\frac{S_1}{U}\right]^{\theta^-}.$$