Pricing Credit Default Swaps with a Random Recovery Rate by a Double Inverse Fourier Transform

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Credit default swaps

Credit risk is an investor's risk of loss arising from a borrower who does not make payments as promised.

The Depository Trust & Clearing Corporation estimates that the size of the global credit derivatives market in 2010 was \$1.66 quadrillion US Dollars. Credit default swaps (CDSs) are the simplest and most popular credit derivatives.

Single-name CDS: A bilateral agreement where the protection buyer transfers the credit risk of a reference entity to the protection seller by paying premiums up to the maturity.

The Lévy first-passage model

Under the risk-neutral setting:

- A firm's asset process $V = \{V_t, t \geq 0\}$ follows

$$V_t = V_0 \, e^{Z_t},$$

  where $Z = \{Z_t, t \geq 0\}$ is a Lévy process with downward jumps.

- $\mathbb{E}(V_t) = V_0 \, e^{rt}$, with r the constant interest rate.

- For a threshold level $L < V_0$, default time is defined as

$$\tau = \inf\{t : V_t \leq L\} = \inf\{t : \ln(V_0/L) + Z_t \leq 0\}.$$

Shifted CMY processes

We assume

$$Z_t = \mu t - S_t$$

with $\mu > 0$ and $S = \{S_t, t \geq 0\}$ from the family of CMY processes with $C, M > 0$ and $0 \leq Y < 1$.

CMY process: the stochastic process that starts at zero and has stationary and independent CMY-distributed increments.

Lévy measure of Z:

$$\Pi(\mathrm{d}x) = Ce^{Mx}(-x)^{-1-Y}\mathrm{d}x, \quad x < 0.$$

Laplace exponent of Z:

$$\psi(s) := \ln \mathbb{E}(e^{sZ_1}) = \mu s + C\Gamma(-Y)\left((M+s)^Y - M^Y\right).$$

- $Y = 0$: Z reduces to a shifted gamma process with

$$\psi(s) = \mu s - C \ln(1 + s/M).$$

- $Y = 0.5$: Z reduces to a shifted inverse Gaussian process with

$$\psi(s) = \mu s - 2\sqrt{\pi} C(\sqrt{s + M} - \sqrt{M}).$$

- Z has paths of infinite jumps and bounded variation.

See Carr *et al.* (2002; *J. of Business*) for properties of the CMY processes.

Remarks on the model

- According to the empirical study by Carr *et al.* (2002; *JB*), risk-neutral processes for equity prices should be processes of infinite activity and finite variation.

- This structural default model was proposed by Madan and Schoutens (2008; *JCR*). It reasonably includes jumps and incorporates skewness in the underlying return distribution.

Random recovery rate

- The CDS has a maturity of T.
- The reference entity defaults at time τ.
- If $\tau \leq T$, the protection seller is required to pay the protection buyer $1 - R_\tau$ for every insured currency unit, where R_τ is the recovery rate when default occurs at τ.
- We assume that R_τ is not fixed. Instead, $R_\tau = R(-X_\tau)$, where $R(\cdot) \in [0, 1]$ is a positive and non-increasing function defined on $[0, \infty)$.

CDS par spread

Let c be the continuously paid CDS spread. The value of the CDS is

$$\underbrace{\mathbb{E}\left[e^{-r\tau}(1-R(-X_\tau))\mathbf{1}_{\{\tau\le T\}}\right]}_{\text{PV of loss leg}} - \underbrace{\mathbb{E}\left[\frac{c}{r}\left(1-e^{-r(\tau\wedge T)}\right)\right]}_{\text{PV of premium leg}}.$$

Then the par spread c is

$$c = \frac{r\mathbb{E}\left[e^{-r\tau}(1-R(-X_\tau))\mathbf{1}_{\{\tau\le T\}}\right]}{\mathbb{E}\left[1-e^{-r(\tau\wedge T)}\right]}.$$

Generalized expected discounted penalty function

Consider the process

$$X_t = x + Z_t, \quad \text{with } x \geq 0.$$

Definition 1: The generalized expected discounted penalty function (EDPF) of X is

$$\phi(x;r) := \mathbb{E}\left[e^{-r\tau} w(-X_\tau, X_{\tau-}, \underline{X}_{\tau-}) \mathbf{1}_{\{\tau<\infty\}} \,\middle|\, X_0 = x \right],$$

and the generalized finite-time EDPF of X is

$$\phi_t(x;r) := \mathbb{E}\left[e^{-r\tau} w(-X_\tau, X_{\tau-}, \underline{X}_{\tau-}) \mathbf{1}_{\{\tau<t\}} \,\middle|\, X_0 = x \right],$$

with $r \geq 0$ and w a bounded measurable function on $\mathbb{R}_+^3 = [0, \infty)^3$.

Biffis and Morales (2010; *IME*) and Kuznetsov and Morales (2014; *SAJ*) have introduced the generalized EDPF into actuarial literature.

Double Laplace transform of $\phi_t(x;r)$

The double Laplace transform of $\phi_t(x;r)$ is defined as

$$g(\lambda, z) = \int_{x=0}^{\infty} \int_{t=0}^{\infty} e^{-\lambda t - zx} \phi_t(x;r) \mathrm{d}t \mathrm{d}x, \quad \lambda, z > 0.$$

Proposition 1: For $r \geq 0$ and $w(-X_\tau, X_{\tau-}, \underline{X}_{\tau-}) = w(-X_\tau)$, $g(\lambda, z)$ has the following formula

$$\begin{aligned} & g(\lambda, z) \\ = & \frac{1}{\lambda(r + \lambda - \psi(z))} \int_{v=0}^{\infty} \int_{u=0}^{\infty} w(v) \Pi(-u - \mathrm{d}v) \left(e^{-zu} - e^{-\psi^{[-1]}(r+\lambda)u} \right) \mathrm{d}u, \end{aligned}$$

where $\psi^{[-1]}(q) = \sup\{s \geq 0 : \psi(s) = q\}, q \geq 0$.

Double inverse Fourier transform

$g(\lambda, z)$ is analytic on the complex plane where $\mathrm{Re}(\lambda), \mathrm{Re}(z) > 0$.

Let $\lambda_1, \lambda_2, z_1, z_2$ be real numbers with $\lambda_1, z_1 > 0$.

$$\begin{aligned}
& g(\lambda_1 - \mathrm{i}\lambda_2, z_1 - \mathrm{i}z_2) \\
= & \int_{x=0}^{\infty} \int_{t=0}^{\infty} \exp\{-\lambda_1 t + \mathrm{i}\lambda_2 t - z_1 x + \mathrm{i}z_2 x\} \phi_t(x; r) \mathrm{d}t \mathrm{d}x \\
= & \int_{x=0}^{\infty} \int_{t=0}^{\infty} \exp\{\mathrm{i}\lambda_2 t + \mathrm{i}z_2 x\} \exp\{-\lambda_1 t - z_1 x\} \phi_t(x; r) \mathrm{d}t \mathrm{d}x.
\end{aligned}$$

$\Rightarrow$ $g(\lambda_1 - \mathrm{i}\lambda_2, z_1 - \mathrm{i}z_2)$ is the double Fourier transform of $\exp\{-\lambda_1 t - z_1 x\} \phi_t(x; r)$.

By the inverse Fourier transform,

$$\phi_t(x;r) = -\frac{1}{4\pi^2}\int_{\Gamma_1}\int_{\Gamma_2} \exp\{\lambda t + zx\} g(\lambda, z)\mathrm{d}\lambda\mathrm{d}z, \tag{1}$$

$\Gamma_1 = \{\lambda_1 + \mathrm{i}\lambda_2 | \lambda_2 = -\infty \cdots + \infty\}, \Gamma_2 = \{z_1 + \mathrm{i}z_2 | z_2 = -\infty \cdots + \infty\}.$

$$\Gamma_1 \xRightarrow{h(\lambda)=\psi(\lambda/\mu)-r} \Gamma_0$$

$$\begin{aligned}\phi_t(x;r) &= -\frac{1}{4\pi^2}\int_{\Gamma_0}\int_{\Gamma_2} \exp\{\lambda t + zx\} g(\lambda, z)\mathrm{d}\lambda\mathrm{d}z \\ &= -\frac{1}{4\pi^2}\int_{\Gamma_1}\int_{\Gamma_2} h'(\lambda)\exp\{h(\lambda)t + zx\} g(h(\lambda), z)\mathrm{d}\lambda\mathrm{d}z. \end{aligned} \tag{2}$$

The idea is from Rogers (2000; *JAP*).

Proposition 2. Assume that $\psi(\cdot)$, the Laplace exponent of process Z, satisfies the following three conditions: for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 0$,

- C1: $(\psi(s) - \mu s)/s \to 0$ as $|s| \to \infty$,
- C2: $|\psi^{[-1]}(s)| \to \infty$ as $|s| \to \infty$, and
- C3: $\mathrm{Re}(\psi^{[-1]}(s)) > 0$.

Then, altering Γ_1 to contour $\Gamma_0 = \psi(\Gamma_1/\mu) - r$ does not change the value of the Fourier integration in (1).

Now the problem is how to evaluate the r.h.s. of (2).

- Approximate by the following double sum

$$
\begin{aligned}
S_N = {} & \frac{h_1 h_2}{4\pi^2} \sum_{n=-Nl_1}^{Nl_1} \sum_{m=-Nl_2}^{Nl_2} h'(a_1 + \mathrm{i}nh_1) g(h(a_1 + \mathrm{i}nh_1), a_2 + \mathrm{i}mh_2) \\
& \times \exp\{th(a_1 + \mathrm{i}nh_1) + x(a_2 + \mathrm{i}mh_2)\}
\end{aligned}
$$

  with $a_1 = \frac{A_1}{2tl_1}, a_2 = \frac{A_2}{2xl_2}, h_1 = \frac{\pi}{tl_1}, h_2 = \frac{\pi}{xl_2}$.
- Use Euler sum to improve approximation accuracy:

$$
\sum_{k=0}^{K} 2^{-K} \binom{K}{k} S_{N+k}.
$$

- Choudhury *et al.* (1994; *AnAP*) and Rogers (2000; *JAP*) suggested to choose appropriate values of A_1, A_2, l_1, l_2, N and K to control the aliasing error, the round off error, and the truncation error.

Numerical experiments

According to Lando and Mortenson (2005; *JIM*) there are different styles of term structures of CDS spreads:

- Investment grade: the spreads are small and the curve is upward sloping.
- Speculative grade: the spreads are larger and the curve is humped in shape.
- Extremely speculative grade: the spreads are very large and the curve shows a downward sloping.

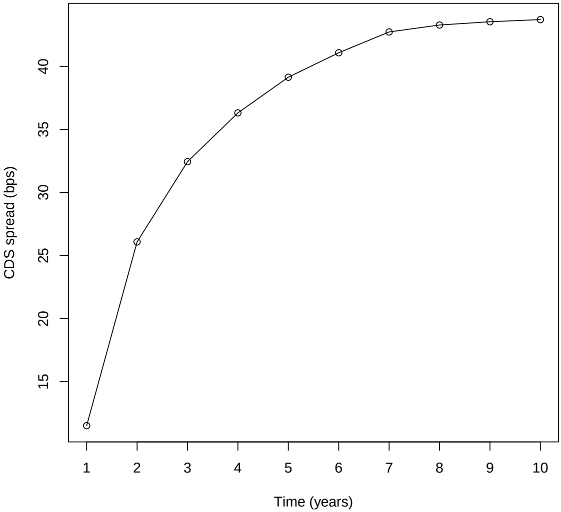

Figure 1: CDS spreads curve assuming that the logarithm of the asset value follows a shifted CMY process with $C = 1, M = 7$, and $Y = 0$

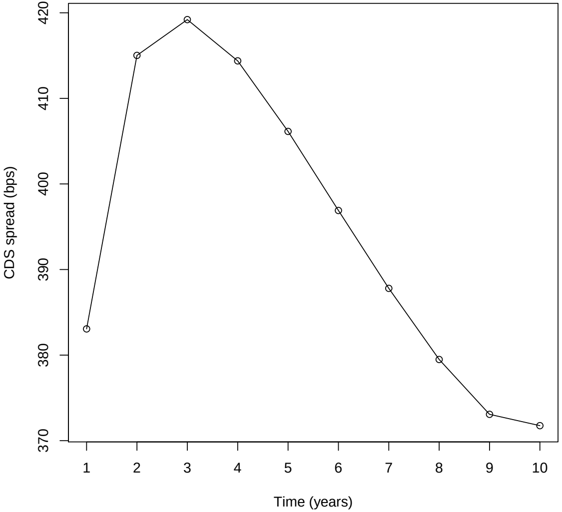

Figure 2: CDS spreads curve assuming that the logarithm of the asset value follows a shifted CMY process with $C = 1$, $M = 3$, and $Y = 0$

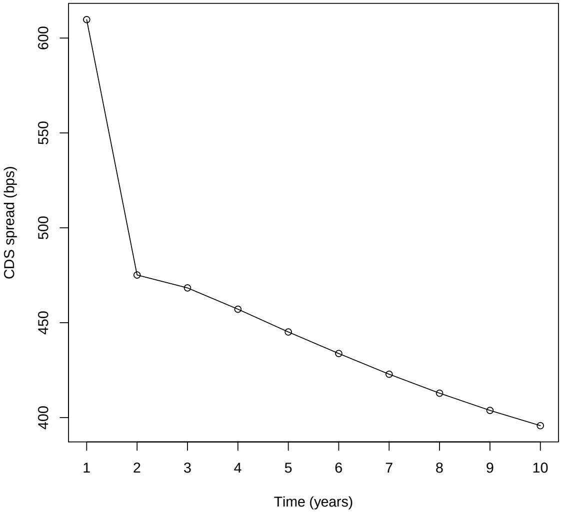

Figure 3: CDS spreads curve assuming that the logarithm of the asset value follows a shifted CMY process with $C = 0.5$, $M = 1.9$, and $Y = 0$

References

- Biffis, E.; Morales, M. (2010). On a generalization of the Gerber-Shiu function to path-dependent penalties. *Insurance: Mathematics & Economics* **46**, no. 1, 92-97.
- Carr, P.; Geman, H.; Madan, D. B.; Yor M. (2002). The fine structure of asset returns: an empirical investigation. *Journal of Business* **75**, no. 2, 305-332.
- Choudhury, G.L.; Lucantoni, D.M.; Whitt, W. (1994). Multidimensional transform inversion with applications to the transient M/G/1 queue. *Annals of Applied Probability* **4**, no. 3, 719-740.
- Kuznetsov, A.; Morales, M. (2014). Computing the finite-time expected discounted penalty function for a family of Lévy risk processes. *Scandinavian Actuarial Journal*, no.1, 1-31.
- Lando, D.; Mortensen, A. (2005). Revisiting the slope of the credit spread curve. *Journal of Investment Management* **3**, no. 4, 1-27.
- Madan, D.; Schoutens, W. (2008). Break on through to the single side. *Journal of Credit Risk* **4**, no. 3, 3-20.
- Rogers, L.C.G. (2000). Evaluating first-passage probabilities for spectrally one-sided Lévy processes. *Journal of Applied Probability* **37**, no. 4, 1173-1180.

Thank you!