Credit risk is an investor’s risk of loss arising from a borrower who does not make payments as promised.

The Depository Trust & Clearing Corporation estimates that the size of the global credit derivatives market in 2010 was $1.66 quadrillion US Dollars. Credit default swaps (CDSs) are the simplest and most popular credit derivatives.

Single-name CDS: A bilateral agreement where the protection buyer transfers the credit risk of a reference entity to the protection seller by paying premiums up to the maturity.
Under the risk-neutral setting:

- A firm’s asset process $V = \{V_t, t \geq 0\}$ follows
  
  $$V_t = V_0 e^{Z_t},$$
  
  where $Z = \{Z_t, t \geq 0\}$ is a Lévy process with downward jumps.

- $\mathbb{E}(V_t) = V_0 e^{rt}$, with $r$ the constant interest rate.

- For a threshold level $L < V_0$, default time is defined as
  
  $$\tau = \inf \{t : V_t \leq L\} = \inf \{t : \ln(V_0/L) + Z_t \leq 0\}.$$
We assume

\[ Z_t = \mu t - S_t \]

with \( \mu > 0 \) and \( S = \{S_t, t \geq 0\} \) from the family of CMY processes with \( C, M > 0 \) and \( 0 \leq Y < 1 \).

**CMY process**: the stochastic process that starts at zero and has stationary and independent CMY-distributed increments.

**Lévy measure of** \( Z \):

\[ \Pi(dx) = Ce^{Mx}(-x)^{-1-Y}dx, \quad x < 0. \]

**Laplace exponent of** \( Z \):

\[ \psi(s) := \ln \mathbb{E}(e^{sZ_1}) = \mu s + C \Gamma(-Y) \left((M+s)^Y - M^Y\right). \]
\( Y = 0: Z \) reduces to a **shifted gamma process** with 

\[ \psi(s) = \mu s - C \ln(1 + s/M). \]

\( Y = 0.5: Z \) reduces to a **shifted inverse Gaussian process** with 

\[ \psi(s) = \mu s - 2\sqrt{\pi}C(\sqrt{s + M} - \sqrt{M}). \]

\( Z \) has paths of infinite jumps and bounded variation.

See **Carr et al. (2002; J. of Business)** for properties of the CMY processes.
Remarks on the model

- According to the empirical study by Carr et al. (2002; JB), risk-neutral processes for equity prices should be processes of infinite activity and finite variation.

- This structural default model was proposed by Madan and Schoutens (2008; JCR). It reasonably includes jumps and incorporates skewness in the underlying return distribution.
The CDS has a maturity of $T$.

The reference entity defaults at time $\tau$.

If $\tau \leq T$, the protection seller is required to pay the protection buyer $1 - R_\tau$ for every insured currency unit, where $R_\tau$ is the recovery rate when default occurs at $\tau$.

We assume that $R_\tau$ is not fixed. Instead, $R_\tau = R(-X_\tau)$, where $R(\cdot) \in [0,1]$ is a positive and non-increasing function defined on $[0,\infty)$.
Let $c$ be the continuously paid CDS spread. The value of the CDS is

$$
\mathbb{E} \left[ e^{-r\tau} (1 - R(-X\tau)) \mathbbm{1}_{\{\tau \leq T\}} \right] - \mathbb{E} \left[ \frac{c}{r} \left( 1 - e^{-r(\tau \wedge T)} \right) \right].
$$

This is the present value of the loss leg minus the present value of the premium leg.

Then the par spread $c$ is

$$
c = \frac{r \mathbb{E} \left[ e^{-r\tau} (1 - R(-X\tau)) \mathbbm{1}_{\{\tau \leq T\}} \right]}{\mathbb{E} \left[ 1 - e^{-r(\tau \wedge T)} \right]}.
$$
Generalized expected discounted penalty function

Consider the process

\[ X_t = x + Z_t, \quad \text{with } x \geq 0. \]

**Definition 1:** The generalized expected discounted penalty function (EDPF) of \( X \) is

\[ \phi(x; r) := \mathbb{E} \left[ e^{-r\tau} w(-X_\tau, X_{\tau-}, X_{\tau-}) 1_{\{\tau < \infty\}} \bigg| X_0 = x \right], \]

and the generalized finite-time EDPF of \( X \) is

\[ \phi_t(x; r) := \mathbb{E} \left[ e^{-r\tau} w(-X_\tau, X_{\tau-}, X_{\tau-}) 1_{\{\tau < t\}} \bigg| X_0 = x \right], \]

with \( r \geq 0 \) and \( w \) a bounded measurable function on \( \mathbb{R}_+^3 = [0, \infty)^3 \).

Biffis and Morales (2010; *IME*) and Kuznetsov and Morales (2014; *SAJ*) have introduced the generalized EDPF into actuarial literature.
The double Laplace transform of $\phi_t(x; r)$ is defined as

$$g(\lambda, z) = \int_0^\infty \int_0^\infty e^{-\lambda t - z x} \phi_t(x; r) \, dt \, dx, \quad \lambda, z > 0.$$ 

**Proposition 1:** For $r \geq 0$ and $w(-X_\tau, X_{\tau-}, X_{\tau-}) = w(-X_\tau)$, $g(\lambda, z)$ has the following formula

$$g(\lambda, z) = \frac{1}{\lambda(r + \lambda - \psi(z))} \int_0^\infty \int_0^\infty w(v) \Pi(-u - dv) \left(e^{-zu} - e^{-\psi^{-1}(r+\lambda)u}\right) \, du,$$

where $\psi^{-1}(q) = \sup \{s \geq 0 : \psi(s) = q\}, q \geq 0$. 
Double inverse Fourier transform

g(\lambda, z) is analytic on the complex plane where Re(\lambda), Re(z) > 0.

Let \lambda_1, \lambda_2, z_1, z_2 be real numbers with \lambda_1, z_1 > 0.

\[
g(\lambda_1 - i\lambda_2, z_1 - iz_2) = \int_{x=0}^{\infty} \int_{t=0}^{\infty} \exp\{-\lambda_1 t + i\lambda_2 t - z_1 x + iz_2 x\} \phi_t(x;r) dt dx
\]
\[
= \int_{x=0}^{\infty} \int_{t=0}^{\infty} \exp\{i\lambda_2 t + iz_2 x\} \exp\{-\lambda_1 t - z_1 x\} \phi_t(x;r) dt dx.
\]

\Rightarrow g(\lambda_1 - i\lambda_2, z_1 - iz_2) is the double Fourier transform of \exp\{-\lambda_1 t - z_1 x\} \phi_t(x;r).
By the inverse Fourier transform,

\[
\phi_t(x; r) = -\frac{1}{4\pi^2} \int_{\Gamma_1} \int_{\Gamma_2} \exp\{\lambda t + zx\} g(\lambda, z) d\lambda dz,
\]

(1)

\[\Gamma_1 = \{\lambda + i\lambda_2 | \lambda_2 = -\infty \cdots + \infty\}, \Gamma_2 = \{z_1 + iz_2 | z_2 = -\infty \cdots + \infty\}.\]

The idea is from Rogers (2000; JAP).
Proposition 2. Assume that $\psi(\cdot)$, the Laplace exponent of process $Z$, satisfies the following three conditions: for $s \in \mathbb{C}$ with $\text{Re}(s) > 0$,

**C1:** $(\psi(s) - \mu s)/s \to 0$ as $|s| \to \infty$,

**C2:** $|\psi^{-1}(s)| \to \infty$ as $|s| \to \infty$, and

**C3:** $\text{Re}(\psi^{-1}(s)) > 0$.

Then, altering $\Gamma_1$ to contour $\Gamma_0 = \psi(\Gamma_1/\mu) - r$ does not change the value of the Fourier integration in (1).
Now the problem is how to evaluate the r.h.s. of (2).

- Approximate by the following double sum

\[ S_N = \frac{h_1 h_2}{4\pi^2} \sum_{n=-Nl_1}^{Nl_1} \sum_{m=-Nl_2}^{Nl_2} h'(a_1 + inh_1) g(h(a_1 + inh_1), a_2 + imh_2) \times \exp\{th(a_1 + inh_1) + x(a_2 + imh_2)\} \]

with \( a_1 = \frac{A_1}{2tl_1}, a_2 = \frac{A_2}{2xl_2}, h_1 = \frac{\pi}{tl_1}, h_2 = \frac{\pi}{xl_2} \).

- Use Euler sum to improve approximation accuracy:

\[ \sum_{k=0}^{K} 2^{-K} \binom{K}{k} S_{N+k}. \]

- Choudhury et al. (1994; AnAP) and Rogers (2000; JAP) suggested to choose appropriate values of \( A_1, A_2, l_1, l_2, N \) and \( K \) to control the aliasing error, the round off error, and the truncation error.
According to Lando and Mortensen (2005; *JIM*) there are different styles of term structures of CDS spreads:

- **Investment grade**: the spreads are small and the curve is upward sloping.
- **Speculative grade**: the spreads are larger and the curve is humped in shape.
- **Extremely speculative grade**: the spreads are very large and the curve shows a downward sloping.
Figure 1: CDS spreads curve assuming that the logarithm of the asset value follows a shifted CMY process with $C = 1$, $M = 7$, and $Y = 0$
Figure 2: CDS spreads curve assuming that the logarithm of the asset value follows a shifted CMY process with $C = 1$, $M = 3$, and $Y = 0$. 
Figure 3: CDS spreads curve assuming that the logarithm of the asset value follows a shifted CMY process with $C = 0.5$, $M = 1.9$, and $Y = 0$. 


Thank you!