Exchangeable risks in actuarial science and quantitative risk management

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i. Motivations

- The following context serves as a motivation
  - We consider a portfolio of homogeneous credit risks
  - According to S&P's ratings, for risks with credit rating of single B :
    - Probability of default = 0.049
    - Pearson's correlation coeffcient between the occurences of two risks = 0.00156
  - Is this correlation negligible ?
  - Can we assume that the risks are independent ?
  - If we ignore it, does it have an impact on the riskiness of the portfolio ?
- To answer these questions :
  - We use the concept of sequence of exchangeable random variables
  - We consider an extension of the classical discrete-time risk model, with exchangeability
- This talk involves two important contributions by Bruno De Finetti :
  - Representation Theorem for sequence of exchangeable random variables
  - Classical discrete-time risk model
- The obtained results can be applied in various contexts

ii. Content

1. A brief historical parenthesis on Bruno De Finetti

- Actuary, Probabilist, Statistician (June 13, 1906 - July 20, 1985)

- Actuary at one of world's largest insurance company : Assicurazioni Generali

- Chair on "Financial Mathematics" in Trieste University
- Chair on "Financial Mathematics" and a Chair on "Calculus of Probabilities" in Sapienza University of Rome
- He has made significant contributions on probability, statistic and risk theory

2. Sequence of exchangeable rvs

- Let $\underline{X} = \{X_k, k \in \mathbb{N}^+\}$ be a sequence of rvs

Definition

$\underline{X}$ is said to be sequence of exchangeable rvs if

$$\left(X_{\sigma(1)}, X_{\sigma(2)}, ..., X_{\sigma(k)}\right) \sim (X_1, X_2, ..., X_k),$$

for $k \in \{2, 3, ...\}$ and for any permutation

$$\left(X_{\sigma(1)}, X_{\sigma(2)}, ..., X_{\sigma(k)}\right)$$

of

$$(X_1, X_2, ..., X_k)$$

2. Sequence of exchangeable rvs

- Representation Theorem (De Finetti).
  - Let $\underline{X} = \{X_k, k \in \mathbb{N}^+\}$ be a sequence of exchangeable rvs
  - The cdf of $(X_1, X_2, ..., X_k)$ can be represented as

$$F_{X_1,...,X_k}(x_1, ..., x_k) = \int F_{X_1,...,X_k|\Theta=\theta}(x_1, ..., x_k)\, \mathrm{d}F_{\Theta}(\theta),$$

    for $k = 2, 3, ...$
  - The reverse is also true
- The joint distribution of $(X_1, X_2, ..., X_k)$ is defined by a common mixture
- Rv Θ :
  - underlying common mixing rv with cdf F_{Θ}
  - unobservable rv Θ
  - random environment
- Important for the infinite sequence :

$$Cov(X_i, X_j) \geq 0$$

  for any pair (i, j)

- Let $\underline{I} = (I_1, ..., I_n)$ be a vector of Bernoulli exchangeable rvs
- Bernoulli rv I_i : occurrence rv
  - Default of ith risk $\Rightarrow I_i = 1$
  - No-default $\Rightarrow I_i = 0$
- Let Θ be a mixing rv defined on $[0,1]$, with cdf F_Θ.
- Conditional joint pmf of $\underline{I}$:

$$\Pr(I_1 = i_1, ..., I_n = i_n | \Theta = \theta) = \prod_{j=1}^{n} \theta^{i_j} (1-\theta)^{1-i_j},$$

  for $(i_1, ..., i_n) \in \{0,1\}^n$ and $\theta \in]0,1[$
- Important application in QRM : "One Factor Bernoulli Risk Model" for homogeneous credit risks
- See e.g. Joe (1997), Cossette et al. (2002), McNeil et al. (2005), Cousin & Laurent (2008)

- Rv Θ : induces a dependence relation between the occurrence rvs
- When $\theta\uparrow$, $\Pr(I_j = 1|\Theta = \theta) = \theta \uparrow$, for all $j = 1, 2, ..., n$
- Definition :

$$\zeta_k = \Pr(I_1 = 1, ..., I_k = 1)$$

for $k = 1, 2, ..., n$

- We have

$$\begin{aligned}\zeta_k &= \int_0^1 \Pr(I_1 = 1, ..., I_k = 1|\Theta = \theta)\,\mathrm{d}F_\Theta(\theta) = \int_0^1 \theta^k \mathrm{d}F_\Theta(\theta) \\ &= E\left[\Theta^k\right]\end{aligned}$$

- Covariance : $Cov(I_1, I_2) = \zeta_2 - \zeta_1^2$
- Pearson's correlation coeffcient : $\rho_P(I_1, I_2) = \frac{\zeta_2 - \zeta_1^2}{\zeta_1 - \zeta_1^2} \in [0, 1]$

- Let us focus on the nb of defaults for the portfolio : rv $N_n = \sum_{j=1}^{n} I_j$
- Conditional pmf of N_n :

$$\Pr(N_n = k \mid \Theta = \theta) = \binom{n}{k} \theta^k (1-\theta)^{(n-k)}$$

(pmf of the binomial distribution)

- Unconditional pmf of N_n :

$$\begin{aligned} \Pr(N_n = k) &= \int_0^1 \Pr(N = k \mid \Theta = \theta) \mathrm{d}F_\Theta(\theta) \\ &= \binom{n}{k} \int_0^\infty \theta^k (1-\theta)^{(n-k)} \mathrm{d}F_\Theta(\theta) \\ &= \binom{n}{k} \sum_{j=0}^{n-k} \binom{n-k}{j} (-1)^j \zeta_{k+j} \end{aligned}$$

(pmf of the mixed-binomial distribution)
(see e.g. Bowman and George (1995), Cossette et al. (2002))

- We want to find the distribution of N_n in order to derive risk quantities related to N_n
- First, how to model $(I_1, ..., I_n)$?
- Several possible approaches were considered
- Two approaches considered in the literature :
  - Approach #1 : Assume a distribution for Θ (e.g. Beta distribution)
  - Approach #2 : Use exchangeable copulas (e.g. Clayton, Frank, Gumbel, etc.)
- In this section, we propose another approach based only on the partial information available

- Our proposed approach :
  - We derive moment bounds on risk quantities related to N_n
  - We assume m known moments for Θ ($\Rightarrow$ m known moments for N_n)
  - **Important :** no distribution is specified for Θ
  - **Equivalently** : no joint distribution is specified for $(I_1, ..., I_n)$
  - Inspired from results obtained by Courtois and Denuit (2009)

- Basic definitions for the approach :
  - We assume that the first m moments of Θ are fixed :

$$E\left[\Theta\right] = \mu_j, \text{ for } j = 1, ..., m$$

  - Let $A_n = \{0, 1, 2, ..., n\}$ be the support of N_n
  - Let $B_l = \left\{\frac{0}{l}, \frac{1}{l}, \frac{2}{l}, ..., \frac{l}{l}\right\}$ be the support of Θ
  - Let $\mathcal{D}\left(\zeta_1, ..., \zeta_m; B_l\right)$ be the class of all rvs Θ with support B_l
  - SL premium :
    $\pi_\Theta\left(\frac{k}{l}\right) = E\left[\max\left(\Theta - \frac{k}{l}; 0\right)\right] = \sum_{j=k}^{\infty}\left(1 - F_\Theta\left(\frac{j}{l}\right)\right)$, for $\frac{k}{l} \in B_l$
  - Let $\mathcal{N}_n$ be the class of all mixed-binomial rvs N_n defined with $\Theta \in \mathcal{D}\left(\zeta_1, ..., \zeta_m; B_l\right)$
  - There is a one-to-one relation between Θ and N_n

- General steps of the approach :
  - Step 1 of 5 : For each $\frac{k}{l} \in B_l$, find the minimal $\pi_{m,\min}\left(\frac{k}{l}\right)$ and the maximal $\pi_{m,\max}\left(\frac{k}{l}\right)$ values of SL premiums such that

$$\pi_{m,\min}\left(\frac{k}{l}\right) \leq \pi_{\Theta}\left(\frac{k}{l}\right) \leq \pi_{m,\max}\left(\frac{k}{l}\right)$$

    for all $\Theta \in \mathcal{D}\left(\zeta_1, ..., \zeta_m; B_l\right)$
  - To find those values, we walk on the "extermal points" of the space $\mathcal{D}\left(\zeta_1, ..., \zeta_m; B_l\right)$
  - Courtois and Denuit (2009) gives the expressions of $\pi_{m,\min}$ and $\pi_{m,\max}$, for $m = 2, 3$

- General steps of the approach :
  - Step 2 of 5 : Define two rvs $\Theta^{(m,\min)}$ and $\Theta^{(m,\max)}$ whose cdfs

$$F_{\Theta^{(m,\min)}} \text{ and } F_{\Theta^{(m,\max)}}$$

are derived from

$$\pi_{m,\min} \text{ and } \pi_{m,\max}$$

with

$$\begin{aligned} F_{\Theta^{(m,\min)}}\left(\frac{k}{l}\right) &= \pi_{m,\min}\left(\frac{k}{l}\right) - \pi_{m,\min}\left(\frac{k+1}{l}\right) \\ F_{\Theta^{(m,\min)}}\left(\frac{k}{l}\right) &= \pi_{m,\min}\left(\frac{k}{l}\right) - \pi_{m,\min}\left(\frac{k+1}{l}\right) \end{aligned}$$

for $k = 0, 1, 2, ..., l-1$. Also, $F_{\Theta^{(m,\min)}}(1) = F_{\Theta^{(m,\min)}}(1) = 1$

- General steps of the approach :
  - Step 3 of 5 : It implies

$$\Theta^{(m,\min)} \preceq_{icx} \Theta \preceq_{icx} \Theta^{(m,\min)}$$

for all

$$\Theta \in \mathcal{D}\left(\zeta_1, ..., \zeta_m; B_l\right),$$

where "$\preceq_{icx}$" $=$ increasing convex order

- General steps of the approach :
  - Step 4 of 5 : When

$$\Theta^{(m,\min)} \preceq_{icx} \Theta \preceq_{icx} \Theta^{(m,\min)}$$

    for all

$$\Theta \in \mathcal{D}\left(\zeta_1, ..., \zeta_m; B_l\right),$$

    it implies that

$$N_n^{(m,\min)} \preceq_{icx} N_n \preceq_{icx} N_n^{(m,\min)}$$

    for all $N_n \in \mathcal{N}_n$
  - $N_n^{(m,\min)}$ is defined by $\Theta^{(m,\min)}$
  - $N_n^{(m,\max)}$ is defined by $\Theta^{(m,\max)}$

- General steps of the approach :
  - Step 5 of 5 : From Denuit et al. (2005), it follows that

$$TVaR_{\kappa}\left(N_n^{(m,\min)}\right) \leq TVaR_{\kappa}\left(N_n\right) \leq TVaR_{\kappa}\left(N_n^{(m,\max)}\right)$$

for all $\kappa \in (0,1)$ and for all

$$N_n \in \mathcal{N}_n$$

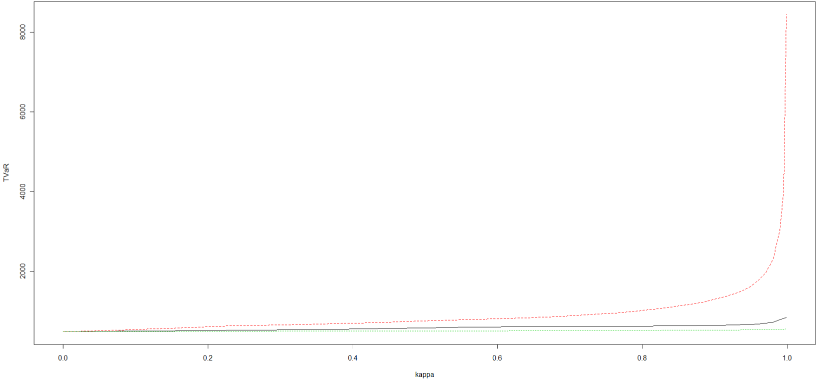
- Additional comments :
  - $E\left[N_n^{(m,\min)}\right] = E\left[N_n^{(m,\max)}\right] = E\left[N_n\right] = n\zeta_1$

- Numerical Illustration with $m = 2$ fixed moments
- We consider an homogeneous portfolio with $n = 10000$ risks
- Probabilities : $\zeta_1 = 0.049; \qquad \zeta_2 = 0.00313$
  - Pearson's correlation coeffcient : $\rho_P(I_1, I_2) = 0.0156$
  - $E[N_{1000}] = 490$
  - $Var(N_{10000}) = 73359$ (vs variance under independence $= 466$)
  - $CV(N_{10000}) = \frac{\sqrt{73359}}{490} = 0.55$ (independence $\Rightarrow \frac{\sqrt{466}}{490} = 0.04$)
- $\Theta \in A_{50} = \left\{0, \frac{1}{50}, \frac{2}{50}, ..., \frac{50}{50}\right\}$
- Source : Real data from 20 years of Standard & Poor's default data (see Table 8.6, page 365, in McNeil et al. (2005))

- Values of $TVaR_{\kappa}\left(N_n^{(m,\min)}\right)$, $TVaR_{\kappa}\left(N_n^{(m,\max)}\right)$ and $TVaR_{\kappa}\left(N_n^{(ind)}\right)$ with $N_n^{(ind)} \sim Binom\left(n, q\right)$

κ	$TVaR_{\kappa}\left(N_n^{(ind)}\right)$	$TVaR_{\kappa}\left(N_n^{(m,\min)}\right)$	$TVaR_{\kappa}\left(N_n^{(m,\max)}\right)$
0	490	490	490
0.5	507.22	584.06	759.66
0.9	528.22	648.64	1299.40
0.99	548.45	794.02	3015.69
0.995	553.54	820.51	4054.77
0.999	564.21	847.25	8439.65

- Values of $TVaR_\kappa\left(N_n^{(m,\min)}\right)$, $TVaR_\kappa\left(N_n^{(m,\max)}\right)$ and $TVaR_\kappa\left(N_n^{(ind)}\right)$ with $N_n^{(ind)} \sim Binom\,(n, q)$



- It is possible to consider more than 2 moments
- We can add the assumption of unimodality for Θ
- The approach can be adapted to derive bounds on VaR

- In this section, we introduce exchangeability in a special case of the classical discrete-rime risk model
- It leads to an application of ruin theory for large portfolios of exchangeable risks

- Proposed by De Finetti (1957)
- Title of his paper : "Su un'impostazione alternativa della teoria collettiva del rischio"
- In English : "An alternative approach in the theory of collective risk"
- Presented at the International Congress of Actuaries
- A standard model in risk theory (see e.g. Bühlmann (1970) and Dickson (2005) for details)

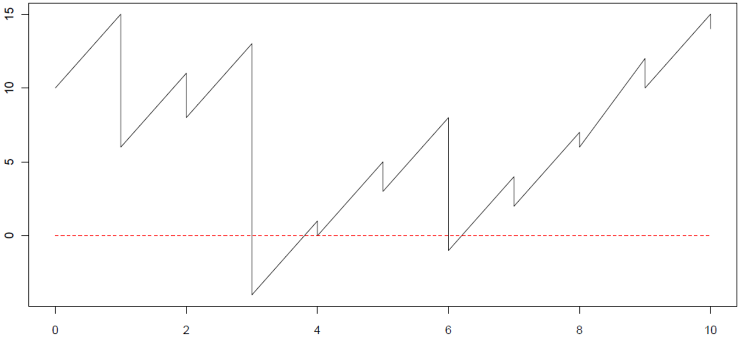
- We consider a portfolio of an insurance company or any financial institution
- $\underline{W} = \{W_k, k \in \mathbb{N}^+\}$: sequence of iid rvs
- Rv W_k : aggregate claim amount in period $k \in \mathbb{N}^+$
- $\pi = (1+\eta) E[W]$: premium income per period
- Rv $L_k = (W_k - \pi)$ = net loss in period $k \in \mathbb{N}^+$
  - $L_k > 0$: loss
  - $L_k < 0$: gain
- Strictly positive security margin : $\eta > 0$
- $E[L_k] = E[W_k] - \pi < 0$ (since $\eta > 0$), for $k \in \mathbb{N}^+$

- Surplus process : $\underline{U} = \{U_k, k \in \mathbb{N}\}$
  - U_k = surplus level at time $k \in \mathbb{N}$
  - $U_0 = u$ = initial surplus
- For $k \in \mathbb{N}^+$: $U_k = U_{k-1} + \pi - W_k = u - \sum_{j=1}^{k} L_j$
- Time of ruin : rv

$$\tau_u = \begin{cases} \inf\limits_{k \in \mathbb{N}^+} \{k, U_k < 0\}, & \text{if } \underline{U} \text{ goes below 0 at least once} \\ \infty, & \text{if } \underline{U} \text{ never goes below 0} \end{cases}$$

- Finite-time ruin probability : $\psi(u, n) = \Pr(\tau_u \leq n)$
- Infinite-time ruin probability : $\psi(u) = \Pr(\tau_u < \infty)$

- A typical sample path of the surplus process $\underline{U}$



- The compound binomial risk model is a special case of the classical discrete-time risk model
- Additional assumptions for the compound binomial risk model :
  - premium income $\pi = 1$
  - $W_k = \begin{cases} X_k, & I_k = 1 \\ 0, & I_k = 0 \end{cases}$
  - $\underline{I} = \{I_k, k \in \mathbb{N}^+\}$: sequence of iid rvs ($I_k \sim I \sim Bern\,(q)$)
  - $\underline{X} = \{X_k, k \in \mathbb{N}^+\}$: sequence of iid rvs ($X_k \sim X \in \mathbb{N}^+$)
  - $\underline{I}$ and $\underline{X}$ are independent
  - initial surplus $u \in \mathbb{N}$
  - references : e.g. Gerber (1988), Shiu (1989), Willmot (1991), DeVylder & Marceau (1996), etc.
- **Extension :**
  - $\underline{I}$ = sequence of exchangeable rvs

- Rv Θ : common mixing rv with cdf F_Θ
- Given $\Theta = \theta$, $\{I_k | \Theta = \theta, k \in \mathbb{N}^+\}$ = sequence of conditionally independent and id rvs
- Notation:
  - $q_\theta = E[I_k | \Theta = \theta] = \theta$
  - $(I_k | \Theta = \theta) \sim Bern(q_\theta)$
  - $\psi_\theta(u)$: conditional ruin probability given that $\Theta = \theta$
- Recall : $q_\theta \uparrow$ as $\theta \uparrow$
- Consequence : there is a θ^* such that
  - when $\theta > \theta^*$, $q_\theta \times E[X] > \pi = 1 \Longrightarrow \psi_\theta(u) = 1$, for all $u \in \mathbb{N}$
  - when $\theta < \theta^*$, $q_\theta \times E[X] < \pi = 1 \Longrightarrow \psi_\theta(u)$ can be computed recursively

- When $\theta < \theta^*$, recursive relation for ψ_θ :
  - $u = 0$: $\psi_\theta(0) = \frac{q_\theta E[X] - q_\theta}{1 - q_\theta}$
  - $u \in \mathbb{N}^+$: $\psi_\theta(u) = \frac{\psi_\theta(u-1) - q_\theta \sum_{j=1}^{u} \psi_\theta(u-j) f_X(j) - q_\theta \overline{F}_X(u+1)}{1 - q_\theta}$
- $\psi(u)$: unconditional ruin probability

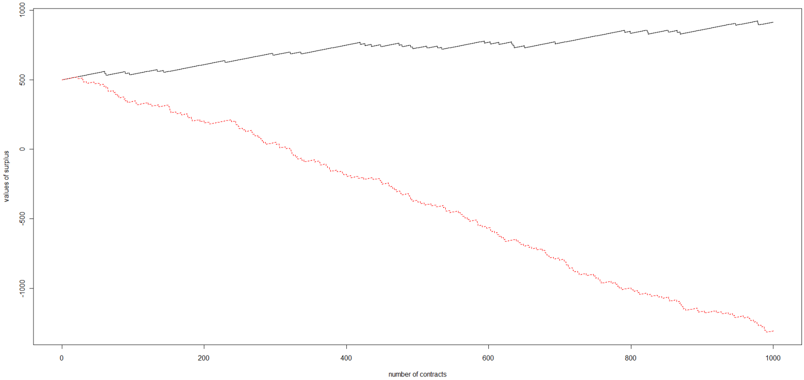
$$\begin{aligned} \psi(u) &= \int_0^\infty \psi_\theta(u) \, \mathrm{d}F_\Theta(\theta) \\ &= \int_0^{\theta^*} \psi_\theta(u) \, \mathrm{d}F_\Theta(\theta) + \overline{F}_\Theta(\theta^*) \end{aligned}$$

- The model is also called the "Mixed compound binomial risk model" (Cossette et al. (2004))

- We propose to apply the CB risk model with exchangeability for an homogeneous portfolio of n credit risks
- As mentioned earlier, $\psi(u, n) =$ ruin probability where n is the number of periods
- Here, n is assumed to be the number of risks
- We assume that the size n of the portfolio is huge ($n \longrightarrow \infty$)
- It implies that we consider the computation of $\psi(u)$
- We use ruin theory to illustrate the dangerousness associated with a huge homogeneous portfolio of credit risks
- See e.g. Seal (1974) for a similar approach for a life insurance portfolio of independent risks ($n =$ nb of contracts)

- Bernoulli rv I_i : occurrence rv for ith risk
  - Default $\Rightarrow I_i = 1$
  - No-default $\Rightarrow I_i = 0$
- Loss rv $L_i = W_i - \pi = W_i - 1$
- Assumption : when default, complete loss $(X_i = b)$
- Interpretation :
  - At time 0, a loan of $b - 1$ is issued to an entity
  - This entity has to reimburse b at time 1
  - At time 1, if no default, b is reimbursed $\Rightarrow$ net loss $= L_i = -1$ (gain)
  - At time 1, if default, 0 is reimbursed $\Rightarrow$ net loss $= L_i = b - 1$ (loss)
- Condition : $\pi = 1 > q \times b = (\text{prob of default}) \times b$

- Two paths of the surplus process (given $\Theta = \theta$)
- Black : $\theta < \theta^*$
- Red : $\theta > \theta^*$



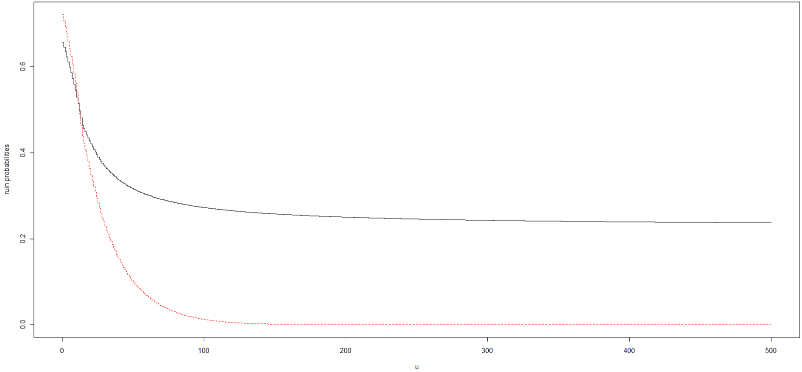
- We compute $\psi(u)$ with $\Theta \sim Beta(\alpha, \beta)$
- Real data from 20 years of Standard & Poor's default data (see Table 8.6, page 365, in McNeil et al. (2005))
- Probabilities for S&P's rating B:
  - $q = \zeta_1 = 0.049; \qquad \zeta_2 = 0.00313$
  - $\rho_P(I_1, I_2) = \frac{0.00313 - 0.049^2}{0.049 - 0.049^2} = 0.0156$
- $\Longrightarrow \Theta \sim Beta(\alpha, \beta)$ with $\alpha = 3.08$ and $\beta = 59.8$

- Recall : $b \times q < \pi = 1$
- Three cases :
  - #1: If $b = 15 \Rightarrow 15 \times 0.049 = 0.735 < 1$ (loan $= 14$)
  - #2: If $b = 18 \Rightarrow 18 \times 0.049 = 0.882 < 1$ (loan $= 17$)
  - #3: If $b = 20 \Rightarrow 20 \times 0.049 = 0.98 < 1$ (loan $= 19$)
- If the credit risks were independent, we should expect that $\psi(u)$ tends to 0 as $u \uparrow \infty$
- However, since the (credit) risks are exchangeable, we will see that $\psi(u)$ will not tend to 0 as $u \uparrow \infty$

- Case #1 : $b = 15$
- Values : ruin probabilities

Initial capital u	$\psi(u)$ (exch.)	$\psi(u)$ (indep)
0	0.6563	0.7213
50	0.3157	0.0997
100	0.2724	0.0124
200	0.2502	0.0002
500	0.2370	0
∞	0.2301	0

- Figure : Ruin probabilities



- For a huge portfolio ($n \to \infty$) and a large initial capital ($u \to \infty$):

$$\lim_{u \to \infty} \psi(u) = \Pr(\Theta > \theta^*)$$

- Additional results :

case	b	$\Pr(\Theta > \theta^*)$
1	15	0.2301
2	18	0.3491
3	20	0.4295

- Similar results for risks with credit rating of double B or triple C
- For a portfolio of exchangeable risks, diversification based on the assumption of independence is not possible
- Remark : We may consider other ruin related quantities

5. Conclusion

- Questions ?
- Thank you for your attention !

- BB: An obligor rated 'BB' is less vulnerable in the near term than other lower-rated obligors. However, it faces major ongoing uncertainties and exposure to adverse business, financial, or economic conditions, which could lead to the obligor's inadequate capacity to meet its financial commitments.
- B: An obligor rated 'B' is more vulnerable than the obligors rated 'BB', but the obligor currently has the capacity to meet its financial commitments. Adverse business, financial, or economic conditions will likely impair the obligor's capacity or willingness to meet its financial commitments.
- CCC: An obligor rated 'CCC' is currently vulnerable, and is dependent upon favorable business, financial, and economic conditions to meet its financial commitments.
- Cited from S&P (2011).

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