A new approach for stochastic ordering of risks

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Presented at
2014 Actuarial Research Conference
UC Santa Barbara

July 16, 2014

Stochastic ordering

- How to compare two given random variables?
- Extensively studied in actuarial science, operations research, quantitative finance, reliability theory, etc.
- Our focus: actuarial science and quantitative finance.
- This topic has attracted attention from researchers for decades.

Some important references

- A comprehensive reference on the general theory:

  Marshall, A. and Olkin, I. (1979). *Inequalities: Theory of Majorization and Its Applications*, Academic Press, New York.

- Review papers in actuarial science:
  1. Dhaene, J., Denuit, M., Goovaerts, M.J., Kaas, R. and Vyncke, D. (2002 a). The concept of comonotonicity in actuarial science and finance: Theory, *Insurance: Mathematics and Economics*, 31(1), 3-33.
  2. Dhaene, J., Denuit, M., Goovaerts, M.J., Kaas, R. and Vyncke, D. (2002 b). The concept of comonotonicity in actuarial science and finance: Application, *Insurance: Mathematics and Economics*, 31(1), 133-161.

Some important references

- Monographs in actuarial science:

  1. van Heerwaarden, A.E. (1991). *Ordering of Risks: theory and actuarial applications*, Tinbergen Institute Research Series, No. 20, Tinbergen Institute, Amsterdam.

  2. Denuit, M. Dhaene, J., Goovaets, M. and Kaas, R. (2005). *Actuarial Theory for Dependent Risks: Measures, Orders and Models*, Wiley. Hoboken, New Jersey.

Three commonly-used stochastic orders

- We will focus on three popular stochastic orders.

    1. stochastic dominance order;
    2. stop-loss order;
    3. convex order.

Stochastic dominance order

Definition

A random variable X is said to precede another random variable Y in stochastic dominance sense, written as $X \leq_{st} Y$, if and only if

$$S_X(t) \leq S_Y(t), \quad \text{for all } t \in R,$$

where S_X and S_Y are the survival functions of X and Y, respectively.

Stop-loss order

Definition

A random variable X is said to precede another random variable Y in stochastic loss order, written as $X \leq_{sl} Y$, if and only if

$$E[(X-d)_+] \leq E[(Y-d)_+], \quad \text{for all } d \in R.$$

Convex order

Definition

A random variable X is said to precede another random variable Y in convex order, written as $X \leq_{cx} Y$, if and only if $X \leq_{sl} Y$ and $E[X] = E[Y]$.

Our goal

- Many other possible stochastic orders on the set of risks.
- All previous works studied properties of stochastic orders on a case by case basis.
- Our goal: introduce a new approach and a uniform framework.
- Main mathematical tool: Riesz space theory, a branch of functional analysis.

Why Riesz space theory?

- Many stochastic orders are partial orders and the underlying space of random variables display a vector space structure.

- Recall that a *binary relation* $\preceq$ on a non-empty set L is a subset of $L \times L$. A binary relation $\preceq$ on a set L is said to be a *partial order* if it has the following three properties:

  (Reflexivity) $x \preceq x$ for $x \in L$;
  (Antisymmetry) if $x \preceq y$ and $y \preceq x$, then $x = y$;
  (Transitivity) if $x \preceq y$ and $y \preceq z$, then $x \preceq z$.

- For an alternative definition (unnecessary for our purpose), see *Set Theory* (2011) by Kenneth Kunen, College Publications.

Why Riesz space theory?

- A natural choice because Riesz space theory (also called vector lattice theory) studies a broad class of partially ordered vector spaces under the most general framework.

- First introduced into economics in 1990's.

  Aliprantis, C.D., Brown, D. J. and Burkinshaw, O. (1987), Edgewoth equilibria, *Econometrica*, 55, 1109-1137.

- For a detailed list of relevant references, see the references listed in

  Aliprantis, C.D., Brown, D. J. and Burkinshaw, O. (1990), *Existence and Optimality of Competitive Equilibria*, Springer, Berlin, Heidelberg, New York.

- Riesz space theory has not been introduced to actuarial science yet.

Why Riesz space theory approach?

Two potential advantages:

1. Simple and clean proofs of complicated theorems;

2. An opportunity of studying different stochastic orders under a unified framework.

Notations and Setup

- Our discussion will always be based on a given probability space $(\Omega, \mathcal{F}, P)$ unless stated otherwise.

- A *risk* is defined to be a nonnegative random variable with finite mean.

- Two P-a.s. equal random variables will be regarded as equivalent, that is, we consider equivalent classes of P-almost surely equal random variables.

- Define

$$S = \{X \mid X \text{ is a random variable}\},$$

and

$$R = \{X \mid X \text{ is a risk}\}.$$

Notations and Setup

- A partially ordered set L is called a *lattice* if the infimum and supremum of any pair of elements in L exist.

- A real vector space L is called an *ordered vector space* if its vector space structure is compatible with the order structure in a manner such that
  (a) if $x \preceq y$, then $x + z \preceq y + z$ for any $z \in L$;
  (b) if $x \preceq y$, then $\alpha x \preceq \alpha y$ for all $\alpha \geq 0$.

- An ordered vector space is called a *Riesz space* (or a *vector lattice*) if it is also a lattice at the same time.

- For two elements x and y in a Riesz space, $x \vee y$ and $x \wedge y$ denotes $\sup\{x, y\}$ and $\inf\{x, y\}$, respectively.

- A vector subspace of a Riesz space is said to be a *Riesz subspace* if it is closed under the lattice operation $\vee$.

Notations and Setup

- If L is an ordered vector space, $x \in L$ and $A \subset L$, then the notation $x \preceq A$ means $x \preceq y$ for every $y \in A$.
- The notation $A \preceq x$ should be interpreted similarly.
- An ordered vector space L is said to satisfy the *interpolation property* if for every pair of nonempty finite subsets E and F of L there exists a vector $x \in L$ such that $E \preceq x \preceq F$.

The main results

1. Separation theorems for three stochastic orders;

2. Decomposition theorems for three stochastic orders;

3. The countable approximation property of risks.

Separation theorems

- Separation theorems for stochastic orders have appeared in various forms in the actuarial literature. For instance,

    Theorem 4.3.1 of van Heerwaarden, A.E. (1991). *Ordering of Risks: theory and actuarial applications*, Tinbergen Institute Research Series, No. 20, Tinbergen Institute, Amsterdam.

    Theorem 3.7 of Müller, A. (1996). Ordering of risks: A comparative study via stop-loss transforms, *Insurance: Mathematics and Economics*, 17, 215-222.

    Proposition 3.4.18 of Denuit, M. Dhaene, J., Goovaets, M. and Kaas, R. (2005). *Actuarial Theory for Dependent Risks: Measures, Orders and Models*, Wiley. Hoboken, New Jersey.

- All existing results only "separate" two risks.

Our new result

Theorem (Separation theorem for stochastic dominance order)

Let R_1 and R_2 be two finite subsets of risks satisfying $X \leq_{st} Y$ for any $X \in R_1$ and $Y \in R_2$. Then there exists a risk Z such that

$$X \leq_{st} Z \leq_{st} Y,$$

for any $X \in R_1$ and $Y \in R_2$. In particular, if X and Y are two risks satisfying $X \leq_{st} Y$, then there exists a risk Z such that

$$X \leq_{st} Z \leq_{st} Y.$$

Proof of the separation theorem

- It is a standard result that every Riesz space has the interpolation property.

  See, for example, Riesz, F. (1940). Sur quelques notions fondamentals dans la theorie générale des opérations linéaires, *Annals of Mathematics*, 41, 174-206.

- Thus, it suffices to show that $(S, \leq_{st})$ is a Riesz space.

- It is easy to see that $(S, \leq_{st})$ is an ordered vector space.

- It remains to show that $\inf\{X, Y\} \in S$ for any pair $X, Y \in S$.

  Caution. The infimum above is taken with respect to the partial order $\leq_{st}$ not the usual order.

Proof of the separation theorem

- Let X and Y be two elements in S. For any $t \in R$, we have

$$S_X(t) \wedge S_Y(t) \leq S_X(t) \quad \text{and} \quad S_X(t) \wedge S_Y(t) \leq S_Y(t),$$

where $\leq$ stands for the usual order on the set of real numbers.

- It follows from the definition of $\leq_{st}$ that $\min\{X, Y\} \leq_{st} \{X, Y\}$.

- This shows that $\min\{X, Y\}$ is a lower bound of $\{X, Y\}$ in $(S, \leq_{st})$.

Proof of the separation theorem

- Next, let $Z \in S$ and $Z \leq_{st} \{X, Y\}$.
- Then $\forall t \in R$,

$$S_Z(t) \leq S_X(t) \quad \text{and} \quad S_Z(t) \leq S_Y(t),$$

that is, $S_Z(t) \leq S_X(t) \wedge S_Y(t)$ for all $t \in R$.

- This shows that $\min\{X, Y\}$ is the greatest lower bound of $\{X, Y\}$ in $(S, \leq_{st})$.
- Therefore, $(S, \leq_{st})$ is a Riesz space.

Remarks

- The above theorem "separates" two subsets of risks.
- Note that $(R, \leq_{st})$ is not an ordered vector space. We often need to tactically consider $(S, \leq_{st})$.
- If we replace $\leq_{st}$ by any partial order, such as the stop-loss order or the convex order, the proof will go through. This yields a separation theorem for any partial order on R.
- The original proof of the separation theorem for the convex order is quite lengthy; our new proof is short and elegant and our result is stronger. This shows the advantage of this Riesz space approach.

Decomposition theorems for three risk orders

Theorem (Decomposition theorem for stochastic dominance order)

If $X, Y_1, ..., Y_n$ *are risks and* $X \leq_{st} Y_1 + ... + Y_n$ *in* $(S, \leq_{st})$*, then there exist risks* $X_1, ..., X_n$ *in* $(S, \leq_{st})$ *such that* $X_i \leq_{st} Y_i, (i = 1, ..., n)$ *and*

$$X = X_1 + ... + X_n.$$

Theorem (Decomposition theorem for stop-loss order)

If $X, Y_1, ..., Y_n$ *are risks and* $X \leq_{sl} Y_1 + ... + Y_n$ *in* $(S, \leq_{sl})$*, then there exist risks* $X_1, ..., X_n$ *in* $(S, \leq_{sl})$ *such that* $X_i \leq_{sl} Y_i, (i = 1, ..., n)$ *and*

$$X = X_1 + ... + X_n.$$

Decomposition theorems for three risk orders

Theorem (Decomposition theorem for convex order)

If $X, Y_1, ..., Y_n$ are nonnegative risks and $X \leq_{cx} Y_1 + ... + Y_n$ in $(S, \leq_{cx})$, then there exist risks $X_1, ..., X_n$ in $(S, \leq_{cx})$ such that $X_i \leq_{cx} Y_i, (i = 1, ..., n)$ and

$$X = X_1 + ... + X_n.$$

Proofs of the three decomposition theorems

- All the three decomposition theorems follow from the following standard result.

Theorem

If $(S, \preceq)$ is an ordered vector space, then the following statements are equivalent.

(1) *S has the interpolation property.*

(2) *S has the decomposition property, that is, if $X, Y_1, ..., Y_n$ are nonnegative elements in S and $X \preceq Y_1 + ... + Y_n$, then there exist nonnegative elements $X_1, ..., X_n$ in S such that*

$$X = X_1 + ... + X_n.$$

- Different from traditional approach, here we "killed three birds with one stone!" This is another advantage of this unified approach.

The countable approximation property of risks

- There is a striking (may not be well-known) result in probability theory.

Definition

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\mathcal{H}$ be a non-empty family of random variables. A random variable η is called the **essential infimum** of $\mathcal{H}$ if η satisfies the following conditions:

(i) $\xi \geq \eta$ *a.s.* for all $\xi \in \mathcal{H}$,

(ii) If η' is another random variable satisfying (i) (i.e. $\xi \geq \eta'$ *a.s.* for all $\xi \in \mathcal{H}$) , then $\eta \geq \eta'$ *a.s.*.

The countable approximation property of risks

Theorem

(1) *Let $\mathcal{H}$ be a non-empty family of random variables, then the essential supremum (respectively infimum) of $\mathcal{H}$ exists, and there are at most countably many element $(\xi_n)_n$ of $\mathcal{H}$ such that*

$$\textit{ess sup } \mathcal{H} = \vee_n \xi_n \ (\textit{respectively ess inf } \mathcal{H} = \wedge_n \xi_n). \tag{1}$$

(2) *In addition, if H is closed under the operation $\vee$ (respectively $\wedge$), i.e.*

$$\xi, \eta \in \mathcal{H} \Rightarrow \exists f \in \mathcal{H} \text{ such that } f = \xi \vee \eta \ (\text{respectively } f = \xi \wedge \eta).$$

then the sequence $(\xi)_n$ can be chosen to be monotone increasing (respectively decreasing).

The countable approximation property of risks

- Can this property extended to $(R, \preceq)$?
- If so, any risk under a partial order can be approximated by a sequence of risks!
- We show the answer is affirmative.
- We call such a property the countable approximation property of risks.

The countable approximation property of risks

Theorem (Countable approximation property of risks)

Every risk can be approximated by a sequence of risks under a partial order, in particular, under stochastic dominance order, stop-loss order and convex order.

Preparation for the proof: more terminologies

- A subset M of a Riesz space L is said to be *solid* if $|x| \leq |y|$ and $y \in S$ implies $x \in M$.
- A solid subspace of a Riesz space is called an *ideal*.
- A Riesz space L is said to have the *countable super property*, that is, every subset of S having a supremum contains an at most countable subset having the same supremum.
- It is known that if L is Riesz space having the countable super property, then any ideal in L also have the countable super property.

Proof of the countable approximation property of risks

- We prove the case of stochastic dominance order for convenience.
- Recall that S consists of equivalent classes of P-a.s. equal random variables on $(\Omega, \mathcal{F}, P)$.
- Since a probability measure is finite, S has the countable super property, that is, every subset of S having a supremum contains an at most countable subset having the same supremum. (Not true if we do not consider equivalent class of P-a.s. random variables.)

Zaanen, A.C. (1997). *Introduction to Operator Theory on Riesz spaces*, Springer, Berlin, New York.

Proof of the countable approximation property of risks

- Let $L_1(\Omega, \mathcal{F}, P)$ denote the space of all P-a.s. equivalent classes of integrable random variable.
- If $X \in S$, $Y \in L_1(\Omega, \mathcal{F}, P)$ and $|X| \leq_{sl} |Y|$, then X is integrable too.
- This means $L_1(\Omega, \mathcal{F}, P)$ is an ideal of S.
- Therefore, $L_1(\Omega, \mathcal{F}, P)$ has the countable super property.

Proof of the countable approximation property of risks

- Since $R \subset L_1(\Omega, \mathcal{F}, P)$, a risk X can be approximated by a sequence of elements in $L_1(\Omega, \mathcal{F}, P)$.

- But X is nonnegative; we may choose this sequence to be nonnegative, that is, X can approximated by a sequence of risks.

Risk measures and stochastic Orders

- The three stochastic orders are closely related to three risk measures.

1. Stochastic dominance $\leftrightarrow$ Quantiles (also called VaR);
2. Stop loss order $\leftrightarrow$ Tail VaR (TVaR);
3. Convex order $\leftrightarrow$ Conditional tail expectation (CTE).

Stochastic dominance vs. Quantiles

Definition

Let X be random variable with cumulative distribution function (CDF) $F_X(x)$. For any $p \in (0,1)$, the p-quantile (risk measure) of X, denoted by $Q_p(X)$ is defined as

$$Q_p(X) = \inf\{x \mid F_X(x) \geq p\}, \tag{2}$$

where $\inf \phi$ is understood to be $+\infty$. We also define the risk measure $Q_p^+(X)$ as

$$Q_p^+(X) = \sup\{x \mid F_X(x) \leq p\}, \tag{3}$$

where $\sup \phi$ is understood to be $-\infty$.

Stochastic dominance vs. Quantiles

Definition

Let X be random variable with CDF $F_X(x)$. For any $p \in (0,1)$, define

$$g_X(p) = \inf\{x \mid F_X(x) > p\}, \tag{4}$$

where $\inf \phi$ is understood to be $+\infty$. Likewise, define

$$h_X(p) = \sup\{x \mid F_X(x) < p\}, \tag{5}$$

where $\sup \phi$ is understood to be $-\infty$.

Stochastic dominance vs. Quantiles

Theorem

Let X and Y be two random variables. Then $X \leq_{st} Y$ if and only if their corresponding quantiles are ordered in the same manner, that is,

$$X \leq_{st} Y \Longleftrightarrow Q_p(X) \leq Q_p(Y) \quad \forall p \in (0,1).$$

Properties of Quantiles

Theorem

Let X be a random variable with CDF $F_X(x)$ and $Q_p(X), Q_p^+(X), g_X(p), h_X(p)$ are quantities defined in Equations (6)-(5). Then the following statements hold.

(1) $Q_p(X) = \lim_{q\uparrow p} g_X(q) = h_X(p)$;

(2) $Q_p^+(X) = \lim_{q\downarrow p} h_X(q) = g_X(p)$;

(3) $g_X(p)$ *is is nonnegative, non-decreasing and right-continuous;*

(4) $h_X(p)$ *is is nonnegative, non-decreasing and left-continuous;*

(5) $Q_p(X)$*, as a function of p, is nonnegative, non-decreasing, left-continuous with right-hand limit, i.e., càglàd;*

(6) $Q_p^+(X)$*, as a function of p, is nonnegative, non-decreasing, right-continuous with left-hand limit, i.e., càdlàg;*

Properties of Quantiles

(7) $g_X(p) < +\infty$ for any $p \in (0,1)$;

(8) $F_X(p) = \inf\{x \mid g_X(x) > p\} = \sup\{x \mid g_X(x) \leq p\}$;

(9) $F_X(g_X(p-)-) = F_X((g_X(p)-)) \leq p$ and $g_X(F_X(p)) \geq g_X(F_X(p-)) \geq p$ for every $p \in (0,1)$, where $f(t-)$ denotes the left-hand limit of a function f at t.

(10) If X is continuous random variable, then $F_X(g_X(p)) = F_X(g_X(p)-) = p$ for ever $p \in (0,1)$.

Applications to credibility theory

We assume a policyholder had losses in the past i-th experience period $1 \leq i \leq n$ and $X_1, ..., X_n$ are independent identically distributed. We are interested in whether the sample mean $\overline{X}_n$ is stable with respect to a predetermined statistic (e.g. the sample mean $E[X_1]$). Precisely, for a given number r and a probability p, we will assign full credibility if

$$P\left[|\overline{X}_n - \xi| \leq r\xi\right] \geq p.$$

Applications to credibility theory

Let $\sigma^2 = Var(X_1)$. In terms of the number of exposure units n, the full credibility standard can be expressed as

$$n \geq \left(\frac{y_p}{r}\right)^2 \left(\frac{\sigma}{\xi}\right)^2,$$

where

$$y_p = \inf \left\{ y \middle| Pr\left(\left| \frac{\overline{X}_n - \xi}{\sigma/\sqrt{n}} \right| \leq y \right) \geq p \right\}. \tag{6}$$

In this case, we can the theorem to conclude that the full credibility standard is guaranteed to be a real number, that is, the infimum in Equation (6) is finite; moreover, the full credibility standard is a non-decreasing and right-continuous function of p.

The theorem also gives a rigorous justification that y_p satisfies

$$Pr\left(\left|\frac{\overline{X}_n - \xi}{\sigma/\sqrt{n}}\right| \le y_p\right) = p.$$

if X_1 has a continuous distribution.

Stop loss order v.s. TVaR

Definition

The Tail Value-at-Risk (TVaR) of a random variable X at level $p \in (0,1)$, often denoted as $TVaR_p[X]$ is defined as

$$TVaR_p[X] = \frac{1}{1-p} \int_p^1 Q_q(X) dq.$$

Theorem

Let X and Y be two random variables. Then

$$X \leq_{sl} Y \iff TVaR_p[X] \leq TVaR_p[Y], \quad \forall p \in (0,1).$$

Stop loss order v.s. TVaR

Definition

The conditional tail expectation (CTE) of a random variable X at level $p \in (0,1)$, often denoted as $CTE_p[X]$ is defined as

$$CTE_p[X] = E[X \mid X > Q_p(X)].$$

Theorem

Let X and Y be two random variables. Then

$$CTE_p[X] \le CTE_p[Y] \Longrightarrow X \le_{sl} Y, \quad \forall p \in (0,1).$$

The converse is not true.

Future research directions

- How a probability transformation, such as the Wang transformation, on $(R, \preceq)$ is going to affect a given risk?

- Properties of positive operators, such as expectation operators or conditional expectation operators, on $(R, \preceq)$.

Thank you!!!

Questions?