# Optimal Capital Allocation: Mean-Variance Models

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# Capital allocation

Assume we have $n$ risks $X_1, \ldots, X_n$. Then, the aggregate loss is

$$S = \sum_{i=1}^{n} X_i,$$

where this aggregate loss $S$ can be interpreted as:

1. the total loss of a corporate, e.g. an insurance company, with the individual losses corresponding to the losses of the respective business units;
2. the loss from an insurance portfolio, with the individual losses being those arising from the different policies; or
3. the loss suffered by a financial conglomerate, while the different individual losses correspond to the losses suffered by its subsidiaries.

# Capital allocation

$S$ is the aggregate loss faced by an insurance company and $X_i$ the loss of business unit $i$. Assume that the company has already determined the aggregated level of capital and denote this total risk capital by $I$:

$$I = I_1 + I_2 + \ldots + I_n.$$

What is the optimal allocation strategy?

# Allocation formulas

- Haircut allocation
  It is a common industry practice, driven by banking and insurance regulations, to measure stand alone losses by a VaR for a given probability level $p$. Assume that

$$l_i = \frac{l}{\sum_{j=1}^{n} F_j^{-1}(p)} F_i^{-1}(p);$$

- Quantile allocation-Dhaene et al., 2002, IME
  The comonotonic sum is

$$S^c = \sum_{i=1}^{n} F_i^{-1}(U),$$

where $U$ is a uniform random variable on $(0, 1)$. Then,

$$l_i = F_i^{-1}(F_{S^c}(l));$$

# Allocation formulas

- Covariance allocation-Overbeck, 2002

$$I_i = \frac{I}{\mathrm{Var}(S)}\mathrm{Cov}(X_i, S);$$

- CTE allocation

$$I_i = \frac{I}{\mathrm{CTE}_p(S)}\mathrm{E}\left(X_i|S > F_S^{-1}(p)\right),$$

where

$$\mathrm{CET}_p(S) = \mathrm{E}\left(S|S > F_S^{-1}(p)\right).$$

# Optimal capital allocation

Decision criterion: Capital should be allocated such that for each business unit the allocated capital and the loss are sufficiently close to each other.

Dhaene, et al. (2011) proposed the following optimization problem to model the capital allocation problem:

$$\min_{\mathbf{l}\in A} \sum_{i=1}^{n} v_i \mathrm{E}\left[\zeta_i \mathbf{D}\left(\frac{X_i - l_i}{v_i}\right)\right], \ s.t. \sum_{i=1}^{n} l_i = l$$

where $v_i$ are nonnegative real numbers such that $\sum_{i=1}^{n} v_i = 1$, and the $\zeta_i$ are non-negative random variables such that $\mathrm{E}[\zeta_j] = 1$.

- The non-negative real number $v_j$ is a measure of exposure or business volume of the $j$-th unit, such as revenue, insurance premium, etc;
- The terms **D** quantify the deviations of the outcomes of the losses $X_j$ from their allocated capital $K_j$;
- The expectations involve non-negative random variables $\zeta_j$ with $\mathrm{E}[\zeta_j] = 1$ that are used as weight factors to the different possible outcomes $\mathbf{D}\,(X_i - l_i)$.

# Quadratic optimization

$$\mathbf{D}(x) = x^2.$$

Consider the following optimization:

$$\min_{\mathbf{l}\in A} \sum_{i=1}^{n} \mathrm{E}\left[\zeta_i \frac{(X_i - l_i)^2}{v_i}\right], \; s.t. \sum_{i=1}^{n} l_i = l.$$

Then, the optimal solution is (Dhaene, et al., 2002)

$$l_i = \mathrm{E}\left(\zeta_i X_i\right) + v_i \left(l - \sum_{j=1}^{n} \mathrm{E}(\zeta_j X_j)\right), \quad i = 1, \ldots, n.$$

# Convex loss function

In the previous work ( Xu and Hu, 2012, IME, 50, 293-298), we considered how the different capital allocation strategies affect the loss function under the general setup. Specifically, the loss function is defined as

$$\mathbf{L}(\mathbf{l}) = \sum_{i=1}^{n} \phi_i(X_i - l_i), \; \mathbf{l} \in A$$

for some suitable convex functions $\phi_i$, where

$$A = \left\{ (l_1, \ldots, l_n) : \sum_{i=1}^{n} l_i = l \right\}.$$

# Convex loss function

discuss the the following optimisation problem:

$$\min_{\mathbf{l}\in A} \sum_{i=1}^{n} \mathrm{P}\left(\mathbf{L}(\mathbf{l}) \geq t\right), \quad \forall t \geq 0;$$

or equivalently,

$$\min_{\mathbf{l}\in A} \mathrm{E}[\Phi\left(\mathbf{L}(\mathbf{l})\right)],$$

for some increasing function $\Phi$, which could be interpreted as a utility function.

# Motivation for Mean-variance models

However, most of the discussion on this topic in the literature has focused only on the magnitude of the loss function $L$. In practice, we might also be interested in the variability of the loss function $L$. The relevant idea has appeared in the premium calculation, see, for example, Valdez (2005) and Furman and Landsman (2006). Furman and Landsman (2006) used the tail variance risk (TVP) measure estimating the variability along the tails to compute the premium.

$$\mathrm{TVP}_q(X) = \mathrm{TCE}_q(X) + \beta \mathrm{TV}_q(X), \quad \beta \geq 0,$$

where $\mathrm{TCE}_q$ means conditional tail expectation,

$$\mathrm{TCE}_q(X) = \mathrm{E}(X|X > x_q), \quad \mathrm{TV}_q(X) = \mathrm{var}\left(X|X > x_q\right),$$

where $x_q$ is $q$th quantile of $X$ or Value-at-Risk (VaR).

The idea of incorporating the variability with the mean might be traced back to the Mean-Variance framework; see, for example, Steinbach (2001) and Landsman (2010). The mean variance (MV) model uses the Mean-Variance risk measurement

$$\mathrm{MV}(X) = \mathrm{E}(X) + \beta \mathrm{var}(X), \quad \beta \geq 0,$$

which is also known as the expected quadratic utility in finance literature.

# MV models for capital allocation

we propose two new MV models to allocation capitals, which control both magnitude and variability of the loss function $L$ using the quadratic function as the distance measure. More specifically, we consider the following MV model:

$$\mathrm{P1}: \begin{cases} \min_{\boldsymbol{p}\in A}\left\{\alpha\mathbb{E}\left[\sum_{i=1}^{n}(X_i-p_i)^2\right]+(1-\alpha)\mathrm{Var}\left(\sum_{i=1}^{n}(X_i-p_i)^2\right)\right\} \\ A=\{\boldsymbol{p}\in\Re^n: p_1+\cdots+p_n=p\}, \end{cases} \tag{1}$$

where $0\le\alpha\le 1$. This optimization problem might be interpreted as $100\alpha\%$ compromise between the mean and variance.

# MV models for capital allocation

The second MV model follows the idea of traditional MV model:

$$\text{P2}:\begin{cases} \min_{\boldsymbol{p}\in A}\left\{\mathbb{E}\left[\sum_{i=1}^{n}(X_i-p_i)^2\right]+\beta\mathrm{var}\left(\sum_{i=1}^{n}(X_i-p_i)^2\right)\right\}; \\ A=\{\boldsymbol{p}\in\Re^n : p_1+\cdots+p_n=p\}, \end{cases} \tag{2}$$

where $\beta>0$. The optimization problems P1 and P2 are equivalent by setting $\beta=(1-\alpha)/\alpha$. However, Models P1 and P2 have different interpretations.

For Model P1, we have the following result.

## Theorem

If the covariance matrix $\Sigma$ of $(X_1, \ldots, X_n)$ is positively definite, then $\boldsymbol{p}^* = (p_1^*, \ldots, p_n^*)$ is an optimal allocation solution to Problem P1, given by

$$p_i^* = \frac{p - \sum_{k=1}^n \sum_{l=1}^n a_{kl}\delta_l}{\sum_{k=1}^n \sum_{l=1}^n a_{kl}} \sum_{j=1}^n a_{ij} + \sum_{j=1}^n a_{ij}\delta_j, \quad i = 1, \ldots, n,$$

where

$$\delta_i = 4(1-\alpha)\sum_{j=1}^n \sigma_{2,j,i} + 2\alpha\mu_i,$$

$\sigma_{2,j,i} = \mathrm{Cov}(X_j^2, X_i)$, $\mu_i = \mathbb{E}(X_i)$, and $(a_{ij})_{n\times n}$ is the inverse matrix of $\mathrm{A} = 8(1-\alpha)\Sigma + 2\alpha \mathrm{I}_n$, where $\mathrm{I}_n$ is the identity matrix.

Similarly, we give the optimization solution to Problem P2.

## Theorem

*If the covariance matrix $\Sigma$ of $(X_1, \ldots, X_n)$ is positively definite, then $\boldsymbol{p}^* = (p_1^*, \ldots, p_n^*)$ is an optimal allocation solution to Problem P2, given by*

$$p_i^* = \frac{p - \sum_{k=1}^n \sum_{l=1}^n a_{kl}\delta_l}{\sum_{k=1}^n \sum_{l=1}^n a_{kl}} \sum_{j=1}^n a_{ij} + \sum_{j=1}^n a_{ij}\delta_j, \quad i = 1, \ldots, n,$$

*where*

$$\delta_i = 4\beta \sum_{j=1}^n \sigma_{2,j,i} + 2\mu_i,$$

$\sigma_{2,j,i} = \mathrm{Cov}(X_j^2, X_i)$, $\mu_i = \mathbb{E}(X_i)$, *and* $(a_{ij})_{n\times n}$ *is the inverse matrix of* $\mathrm{A} = 8\beta \sum + 2\,\mathrm{I}_n$.

# Some special cases

## Corollary

Let $X_1, \ldots, X_n$ be independent random variables. Then $\boldsymbol{p}^* = (p_1^*, \ldots, p_n^*)$ is an optimal allocation solution to P1, given by

$$
\begin{aligned}
p_i^* &= \mu_i + \frac{1}{4(1-\alpha)\sigma_i^2 + \alpha} \times \\
&\left[ \frac{p - \sum_{j=1}^n \mu_j - \sum_{j=1}^n [2(1-\alpha)\gamma_j \sigma_j^2]/[4(1-\alpha)\sigma_j^2 + \alpha]}{\sum_{j=1}^n 1/[4(1-\alpha)\sigma_j^2 + \alpha]} \right. \\
&\left. +2(1-\alpha)\gamma_i \sigma_i^3 \right],
\end{aligned}
$$

where $\mu_j = \mathbb{E}(X_j)$, and $\gamma_j = \mathbb{E}(X_j - \mu_j)^3/\sigma_j^3$, the skewness of $X_j$.

## Corollary

Let $X_1, \ldots, X_n$ be exchangeable random variables. If $\boldsymbol{p}^* = (p_1^*, \ldots, p_n^*)$ is an optimal allocation solution to Problem P1 (P2), then

$$p_1^* = \cdots = p_n^* = \frac{p}{n}.$$

In fact, if $X_1, \ldots, X_n$ are exchangeable random variables, we may have a general result. First, we recall the concepts of majorization and of Schur-convexity. Let $x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(n)}$ be the increasing arrangement of components of the vector $\boldsymbol{x} = (x_1, \ldots, x_n)$. For vectors $\boldsymbol{x}, \boldsymbol{y} \in \Re^n$, $\boldsymbol{x}$ is said to be majorized by $\boldsymbol{y}$, denoted by $\boldsymbol{x} \preceq_{\mathrm{m}} \boldsymbol{y}$, if

$$\sum_{i=1}^{j} x_{(i)} \geq \sum_{i=1}^{j} y_{(i)} \text{ for } j = 1, \ldots, n-1,$$

and $\sum_{i=1}^{n} x_{(i)} = \sum_{i=1}^{n} y_{(i)}$. A real-valued function $\phi$ defined on a set $A \subseteq \Re^n$ is said to be Schur-convex on $A$ if, for any $\boldsymbol{x}, \boldsymbol{y} \in A$,

$$\boldsymbol{x} \succeq_{\mathrm{m}} \boldsymbol{y} \Longrightarrow \phi(\boldsymbol{x}) \geq \phi(\boldsymbol{y}).$$

For extensive and comprehensive details on the theory of majorization orders and their applications, please refer to Marshall et al. (2011).

## Theorem

Let $X_1, \ldots, X_n$ be exchangeable random variables. If

$$(p_1, \ldots, p_n) \succeq_{\mathrm{m}} (p_1^*, \ldots, p_n^*),$$

then

$$\mathbb{E}\left(\sum_{i=1}^{n}(X_i - p_i)^2\right) \geq \mathbb{E}\left(\sum_{i=1}^{n}(X_i - p_i^*)^2\right),$$

and

$$\mathrm{Var}\left(\sum_{i=1}^{n}(X_i - p_i)^2\right) \geq \mathrm{Var}\left(\sum_{i=1}^{n}(X_i - p_i^*)^2\right).$$

# Centrally symmetric distributions

A random vector $\mathbf{X} = (X_1, \ldots, X_n)$ has a distribution centrally symmetric about $\mu = (\mu_1, \ldots, \mu_n)$ if $\mathbf{X} - \mu$ and $\mu - \mathbf{X}$ have the same distribution. It includes many distribution families, one of which is elliptical distribution family. Elliptical distributions are generalizations of the multivariate normal distributions. The class of elliptical distributions contains many well-known distributions as special cases:

- multivariate normal
- multivariate Cauchy
- multivariate exponential
- multivariate t-distributions

Assume $X_1, \ldots, X_n$ have a distribution centrally symmetric about $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)$. If the covariance matrix $\Sigma$ of $(X_1, \ldots, X_n)$ is positively definite, and $\boldsymbol{p}^* = (p_1^*, \ldots, p_n^*)$ is an optimal allocation solution to Problem P1 (P2), then

$$p_i^* = \frac{p - \sum_{k=1}^n \sum_{l=1}^n a_{kl}\delta_l}{\sum_{k=1}^n \sum_{l=1}^n a_{kl}} \sum_{j=1}^n a_{ij} + \sum_{j=1}^n a_{ij}\delta_j,$$

where $\delta_i = 8(1-\alpha) \sum_{j=1}^n \mu_j \sigma_{j,i} + 2\alpha\mu_i$ $(\delta_i = 8\beta \sum_{j=1}^n \mu_j \sigma_{j,i} + 2\mu_i)$, $\mu_j = \mathrm{E}(X_j)$, and $\sigma_{j,i} = \mathrm{Cov}(X_j, X_i)$.

# Example

We use the data reported in *Panjer* (2002). In that example, an insurance company has 10 lines of business with risks represented by the random vector $\mathbf{X} = (X_1, \ldots, X_{10})$. Table 1 reports optimal allocation strategies for different $\alpha$'s with a total capital $p = 200$. Table 2 reports the case for different $\beta$'s for Problem P2. It might be observed that we have negative allocations which reflect the diversification benefit, in that these lines of business may reduce capital requirements for the the company as a whole.

# P1 with a total capital $p = 200$ and different $\alpha$ values

| $\alpha$ | 0 | .1 | .2 | .3 | .4 | .5 |
|---|---|---|---|---|---|---|
| $p_1^*$ | -39.477 | -17.464 | -6.564 | 0.343 | 5.407 | 9.513 |
| $p_2^*$ | 133.010 | 120.886 | 111.949 | 103.745 | 95.468 | 86.719 |
| $p_3^*$ | 193.612 | 125.578 | 91.796 | 70.746 | 55.914 | 44.620 |
| $p_4^*$ | 21.532 | 21.972 | 23.632 | 25.739 | 28.023 | 30.312 |
| $p_5^*$ | 30.812 | 27.869 | 26.257 | 24.892 | 23.470 | 21.868 |
| $p_6^*$ | 75.951 | 88.926 | 92.671 | 92.691 | 90.543 | 86.746 |
| $p_7^*$ | 23.138 | 28.453 | 30.753 | 31.886 | 32.388 | 32.453 |
| $p_8^*$ | -117.865 | -94.850 | -81.075 | -70.303 | -60.548 | -50.878 |
| $p_9^*$ | -94.977 | -82.504 | -73.940 | -66.437 | -59.034 | -51.215 |
| $p_{10}^*$ | -25.736 | -18.867 | -15.480 | -13.304 | -11.631 | -10.138 |
| $\alpha$ | .6 | .7 | .8 | .9 | 1 | |
| $p_1^*$ | 13.120 | 16.530 | 20.054 | 24.339 | 32.277 | |
| $p_2^*$ | 77.253 | 66.954 | 55.992 | 45.609 | 44.427 | |
| $p_3^*$ | 35.535 | 27.899 | 21.198 | 14.935 | 7.437 | |
| $p_4^*$ | 32.412 | 33.997 | 34.372 | 31.713 | 19.287 | |
| $p_5^*$ | 20.023 | 17.876 | 15.338 | 12.173 | 6.737 | |
| $p_6^*$ | 81.394 | 74.308 | 64.986 | 52.135 | 30.637 | |
| $p_7^*$ | 32.135 | 31.397 | 30.067 | 27.587 | 20.997 | |
| $p_8^*$ | -40.693 | -29.490 | -16.785 | -2.283 | 11.077 | |
| $p_9^*$ | -42.574 | -32.682 | -20.968 | -6.532 | 10.977 | |
| $p_{10}^*$ | -8.605 | -6.790 | -4.254 | 0.323 | 16.147 | |

# P2 with a total capital $p = 200$ and different $\beta$ values

| $\beta$ | .1 | .2 | .3 | .4 | .5 | .6 |
|---|---|---|---|---|---|---|
| $p_1^*$ | 24.825 | 21.340 | 18.933 | 17.018 | 15.398 | 13.980 |
| $p_2^*$ | 44.823 | 52.336 | 59.399 | 65.417 | 70.479 | 74.757 |
| $p_3^*$ | 14.361 | 19.092 | 23.188 | 26.895 | 30.318 | 33.513 |
| $p_4^*$ | 31.175 | 33.980 | 34.457 | 34.148 | 33.557 | 32.875 |
| $p_5^*$ | 11.834 | 14.374 | 16.169 | 17.541 | 18.630 | 19.517 |
| $p_6^*$ | 50.686 | 61.205 | 68.144 | 73.132 | 76.885 | 79.798 |
| $p_7^*$ | 27.251 | 29.422 | 30.560 | 31.250 | 31.696 | 31.994 |
| $p_8^*$ | -0.893 | -12.146 | -20.881 | -27.777 | -33.367 | -38.007 |
| $p_9^*$ | -5.038 | -16.519 | -24.811 | -31.136 | -36.150 | -40.238 |
| $p_{10}^*$ | 0.975 | -3.084 | -5.157 | -6.487 | -7.446 | -8.190 |
| $\beta$ | .7 | .8 | .9 | 1 | 2 | 3 |
| $p_1^*$ | 12.711 | 11.558 | 10.497 | 9.513 | 2.178 | -2.793 |
| $p_2^*$ | 78.410 | 81.560 | 84.305 | 86.719 | 101.019 | 107.822 |
| $p_3^*$ | 36.517 | 39.357 | 42.053 | 44.620 | 65.286 | 80.230 |
| $p_4^*$ | 32.183 | 31.518 | 30.893 | 30.312 | 26.490 | 24.651 |
| $p_5^*$ | 20.255 | 20.877 | 21.408 | 21.868 | 24.433 | 25.570 |
| $p_6^*$ | 82.108 | 83.970 | 85.491 | 86.746 | 92.178 | 93.012 |
| $p_7^*$ | 32.193 | 32.324 | 32.406 | 32.453 | 32.110 | 31.420 |
| $p_8^*$ | -41.935 | -45.317 | -48.269 | -50.878 | -66.996 | -75.478 |
| $p_9^*$ | -43.647 | -46.542 | -49.037 | -51.215 | -63.988 | -70.133 |
| $p_{10}^*$ | -8.795 | -9.305 | -9.747 | -10.138 | -12.710 | -14.300 |

# Multivariate gamma distribution

There are a number of multivariate gamma distributions in the literature. In the following, we consider losses or risks which follow a multivariate gamma distribution introduced by Cheriyan (1941), and generalized by Mathai and Moschopoulos (1991). This distribution has been examined in Furman and Lansman (2005) for risk capital allocations. Let $X_1, \ldots, X_n$ be independent gamma random variables with $X_i \sim \Gamma(\alpha_i, \beta_i)$, $i = 0, \ldots, n$. Denote

$$Y_j = \frac{\beta_0}{\beta_j} X_0 + X_j, \quad j = 1, \ldots, n.$$

The joint distribution of the random vector $\mathbf{Y}^\tau = (Y_1, \ldots, Y_n)$ is the multivariate gamma distribution defined in Mathai and Moschopoulos (1991).

# Example

Assume that a company has three business lines $Y_1$, $Y_2$, $Y_3$, which follow a multivariate gamma distribution defined in Mathai and Moschopoulos (1991), with underlying random variables $(X_0, X_1, X_2, X_3)$. Assume that $(\alpha_0, \alpha_1, \alpha_2, \alpha_3) = (1, 2, 3, 5)$, and $(\beta_0, \beta_1, \beta_2, \beta_3) = (.3, .1, .2, .4)$. Hence, the covariance matrix is

$$\Sigma = \begin{pmatrix} 300 & 50 & 25 \\ 50 & 100 & 12.5 \\ 25 & 12.5 & 37.5 \end{pmatrix}.$$

| $\alpha$ | $p_1^*$ | $p_2^*$ | $p_3^*$ |
|---|---|---|---|
| 0 | 5.405 | 51.654 | 12.941 |
| .1 | 5.458 | 51.643 | 12.899 |
| .2 | 5.525 | 51.629 | 12.846 |
| .3 | 5.610 | 51.612 | 12.778 |
| .4 | 5.724 | 51.588 | 12.688 |
| .5 | 5.882 | 51.556 | 12.563 |
| .6 | 6.118 | 51.506 | 12.376 |
| .7 | 6.507 | 51.423 | 12.070 |
| .8 | 7.273 | 51.256 | 11.472 |
| .9 | 9.468 | 50.745 | 9.787 |
| 1 | 31.667 | 21.667 | 16.667 |

Table : Optimal capital allocations for different $\alpha$'s in Problem P1 with a total capital $p = 70$.

| $\beta$ | $p_1^*$ | $p_2^*$ | $p_3^*$ |
|---|---|---|---|
| .1 | 9.889 | 50.641 | 9.469 |
| .2 | 7.724 | 51.155 | 11.122 |
| .3 | 6.968 | 51.323 | 11.709 |
| .4 | 6.584 | 51.406 | 12.009 |
| .5 | 6.352 | 51.456 | 12.192 |
| .6 | 6.196 | 51.489 | 12.315 |
| .7 | 6.084 | 51.513 | 12.403 |
| .8 | 6.000 | 51.531 | 12.469 |
| .9 | 5.934 | 51.545 | 12.521 |
| 1 | 5.882 | 51.556 | 12.563 |
| 2 | 5.644 | 51.606 | 12.751 |

Table : Optimal capital allocations for different $\beta$'s in Problem P2 with a total capital $p = 70$.

# Discussion

It is of interest to consider the absolute deviation distance measure. For example, one may consider the following MV model:

$$\mathrm{P3}: \begin{cases} \min_{\boldsymbol{p}\in A}\left\{\mathbb{E}\left[\sum_{i=1}^{n}|X_i-p_i|\right]+\beta\mathrm{Var}\left(\sum_{i=1}^{n}|X_i-p_i|\right)\right\}; \\ A=\{\boldsymbol{p}\in\Re^n : p_1+\ldots+p_n=p\}, \end{cases}$$

where $\beta \geq 0$. Generally, we have to solve a nonlinear optimization problem for P3. There is no a simple closed form solution in contrast to Problem P1 or P2.

Thank you!