# Forward Mortality Modelling of Multiple Populations

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# Motivation

- Similar populations have similar mortality experience.
- Framework for jointly modelling and forecasting similar populations.
- Forward framework - model for systematic longevity risk.
- Risk management application:
  - Portfolio of annuitants aged 65.
  - Benefits of a cross population survivor swap (static hedge).
  - Reduction in unexpected portfolio cash flows.
  - Reduction is solvency capital requirements.

# Introduction

- Continuous time stochastic framework - two approaches:
  - Short rate model - Affine Term Structure Model (ATSM)
  - Forward rate model - Heath-Jarrow-Morton (HJM)
- Finite dimensional realisation of HJM $\rightarrow$ State-space model
- ATSMs (state-space) pass qualitative modelling criteria:
  - Biological reasonable forecasts and distribution
  - Robustness to historical sample period
  - Parsimonious
  - Transparency
  - Ease of implementation (closed-form likelihood function)
- Forward mortality model:
  - Model for systematic longevity risk.
  - Ideal for risk management and pricing applications.

# Finite Dimensional Realisation (FDR)

- Link between HJM and state-space model (ATSM).
- Requires a deterministic volatility function.
- Greater flexibility than ATSM (ATSM is a special case).
- Common factor model: standard ATSM.
- Dependent factor model: no closed-form solution in ATSM.
- FDR allows greater flexibility in designing a model.

# Finite Dimensional Realisation (FDR)

For a given deterministic volatility function $\sigma(\tau)$, the forward mortality process

$$d\mu(t,\tau) = \left\{\frac{\partial \mu(t,t+\tau)}{d\tau} + \sigma(\tau)\int_0^\tau \sigma(s)' ds\right\}dt + \sigma(\tau)dW^Q(t) \tag{1}$$

has a finite dimensional realisation if the volatility function is of the form

$$\sigma(\tau) = C_0 e^{A\tau} B, \tag{2}$$

then a state-space representation our system is

$$\begin{aligned} dZ(t) &= A^P(\theta^P - Z(t))dt + BdW^P(t), \qquad Z(0) = \psi \\ y(t,\tau) &= \frac{\int_0^\tau C(s)ds}{\tau} Z(t) + \frac{\int_0^\tau \Theta(s)ds}{\tau} + \epsilon(t). \end{aligned} \tag{3}$$

# Dependent Factor Model

- 3-factor model. Interaction between population dependent factors.

$$\sigma_1(\tau) = [\sigma_1 e^{-\delta_1\tau}, \sigma_{12} e^{-\delta_2\tau}, \sigma_3 e^{-\delta_3\tau}]$$
$$\sigma_2(\tau) = [\sigma_{21} e^{-\delta_1\tau}, \sigma_2 e^{-\delta_2\tau}, \sigma_3 e^{-\delta_3\tau}]$$

For the volatility function

$$\sigma(\tau) = C_0 e^{A\tau} B, \tag{4}$$

The finite dimensional realisation is

$$A = -\begin{pmatrix} \delta_{11} & 0 & 0 & 0 & 0 \\ 0 & \delta_{22} & 0 & 0 & 0 \\ 0 & 0 & \delta_{22} & 0 & 0 \\ 0 & 0 & 0 & \delta_{11} & 0 \\ 0 & 0 & 0 & 0 & \delta_{33} \end{pmatrix} \quad B = \begin{pmatrix} \sigma_{11} & 0 & 0 \\ 0 & \sigma_{12} & 0 \\ 0 & \sigma_{22} & 0 \\ \sigma_{21} & 0 & 0 \\ 0 & 0 & \sigma_{33} \end{pmatrix}. \tag{5}$$

# Dependent Factor Model

The real-world process (5 states). 'Essentially Affine'

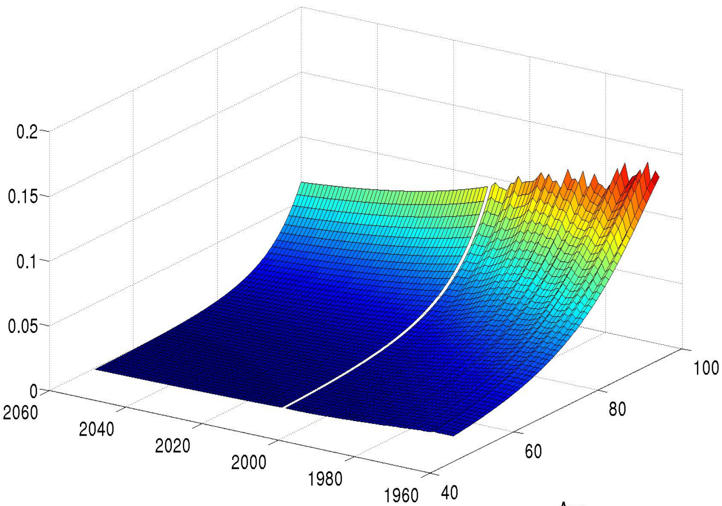
$$dZ(t) = A^P(\theta^P - Z(t))dt + BdW^P(t), \qquad Z(0) = \psi \tag{6}$$

is defined as

$$A^P = -\begin{pmatrix} \kappa_{11} & 0 & 0 & 0 & 0 \\ 0 & \kappa_{22} & 0 & 0 & 0 \\ 0 & 0 & \kappa_{33} & 0 & 0 \\ 0 & 0 & 0 & \kappa_{44} & 0 \\ 0 & 0 & 0 & 0 & \kappa_{55} \end{pmatrix},$$

and we set $\theta^P = (0, 0, 0, 0, 0)'$.

# $y(t, \tau)$ Mortality State-Space Model (Australian Males)



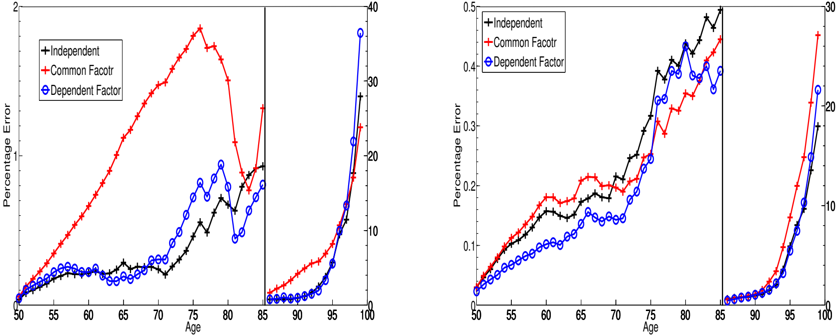
# Fitting Results

- Fit the state-space model with a standard Kalman filter.
- Historical data from the Human Mortality Database.
  - Australian males ages 50 to 99 for the years 1965 to 2009.
  - Swedish males ages 50 to 99 for the years 1965 to 2009.
- Compare with 2-factor independent model.

| Model | Params | Log Likelihood | Aus RMSE | Swe RMSE |
|---|---|---|---|---|
| Independent Aus. | 13 | 14242.41 | 0.00126106 | - |
| Independent Swe. | 13 | 15050.73 | - | 0.00101177 |
| Common Factor | 19 | 27119.67 | 0.00164847 | 0.00149174 |
| Dependent Factor | 19 | 29124.43 | 0.00151793 | 0.00115905 |

Table: Fitting Results

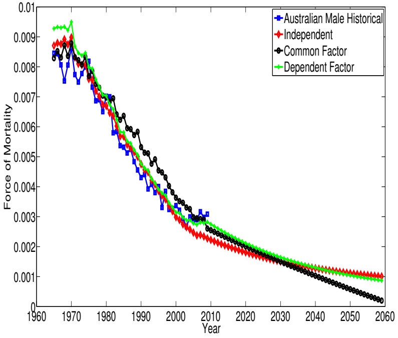
# Fitting Results



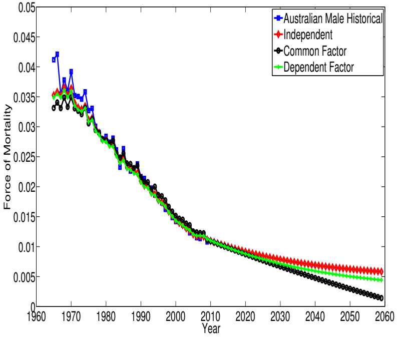

(a) Australian Males (b) Swedish Males

Figure: MAPE

# Fitting Results



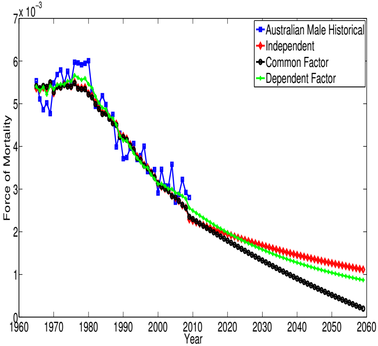

(a) $\mu_{50}(t)$



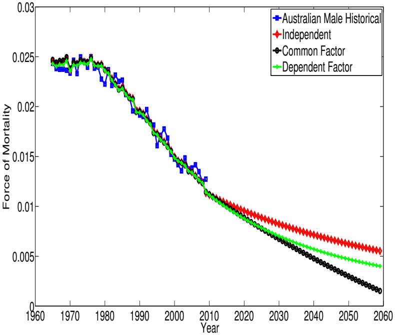

(b) $\mu_{65}(t)$

Figure: Australian Male

# Fitting Results




(a) $\mu_{50}(t)$




(b) $\mu_{65}(t)$

Figure: Swedish Male

# Cohort Survivor Curves

- Deterministic volatility function - Gaussian dynamics.
- Volatility is age dependent (not time dependent).
- Martingale measure is not unique.
- Change of measure required:
  - State-space model - period mortality rates
  - Forward mortality model - cohort mortality rates
- Define $x_0$ the youngest age in the state-space model
- Extract best estimate cohort forecasts from the state-space model.

# Cohort Survivor Curves

- Radon-Nikodym density $\left.\frac{d\widehat{Q}}{dQ}\right|_{F_t} = e^{-\int_0^t \lambda(s)dW_\mu(s) - \frac{1}{2}\int_0^t |\lambda(s)|^2 ds}$
- The mortality process under the cohort measure is

$$\begin{aligned}\mu(t,s;x_0) =& \left[\mu(0,s;x_0) + \int_0^t \sigma_\mu(u,s;x_0)\lambda(u)du\right] + \int_0^t \nu_\mu(u,s;x_0)du \\ &+ \int_0^t \sigma_\mu(u,s;x_0)dW_\mu^{\widehat{Q}}(u) \\ =& \widehat{\mu}(0,s;x_0) + \int_0^t \nu_\mu(u,s;x_0)du + \int_0^t \sigma_\mu(u,s;x_0)dW_\mu^{\widehat{Q}}(u),\end{aligned}$$

where $\widehat{\mu}(0,s;x_0)$ is the best estimate cohort mortality rate

$$\widehat{\mu}(0,s;x_0) = \mu(0,s;x_0) + \int_0^t \sigma_\mu(u,s;x_0)\lambda(u)du. \tag{7}$$

- Note: no change in the volatility function.

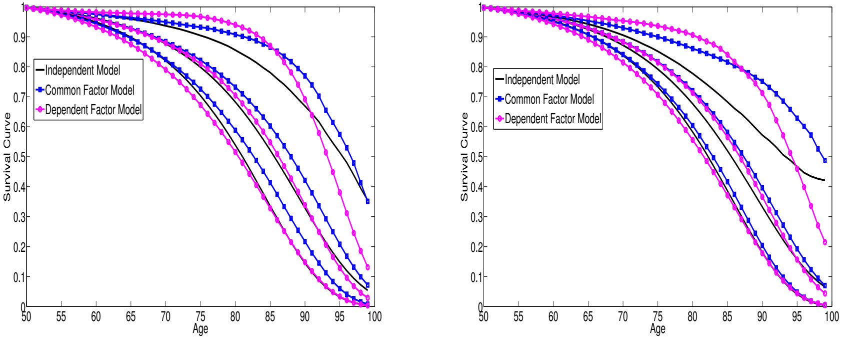
# Cohort Survivor Curves

- For a cohort that is aged $x$ at time-t, we know that $t - (x - x_0)$ years ago this cohort was aged $x_0$.
- We perform the cohort measure change at time-($t - (x - x_0)$).
- Requires a change to the volatility function,

$$\sigma(\tau; x) = C_0 e^{A(\tau + (x - x_0))} B = \widehat{C}_0 e^{A\tau} B, \qquad (8)$$

  where $\widehat{C}_0 = C_0 e^{A(x - x_0)}$.
- Doesn't effect the estimation of the volatility function.

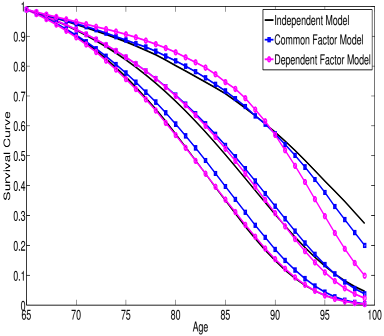
# Fitting Results



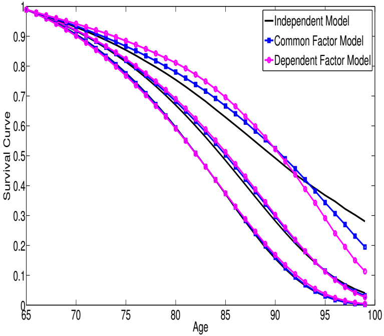

(a) Australian Males (b) Sweish Males

Figure: Forecast Cohort Survivor Curve Distribution - Age 50 in 2010

# Fitting Results




(a) Australian Males




(b) Swedish Males

Figure: Forecast Cohort Survivor Curve Distribution - Age 65 in 2010

# Longevity Risk Management

- Static hedge of longevity risk with a survivor swap.
- Annuity portfolio of Australian males, aged 65 in 2010.
- Survivor swap based on Australian or Swedish male population (index based).
- Swap fixed for floating annuity payment cash flow.
- No longevity risk premium (Martingale is cohort best estimate)
- Australian survivor swap
  - Systematic longevity risk transfer.
  - Idiosyncratic risk.
- Swedish survivor swap
  - Population basis risk.
  - Idiosyncratic risk.
- Discrete HJM Monte Carlo simulation.

# HJM Simulation

- The insurer expected cash flow is

$$S_1(0,t;x_{65}) = e^{-\int_0^t \mu_1(0,s;x_{65})ds}$$

- For each simulation path m, the random death time is

$$\tau_i^{(m)} = \inf\left\{t_i : \sum_{t_s=t_0}^{t_i} \mu_1^{(m)}(t_s;x_{65}) \geq \varrho\right\},$$

- The number of survivors at time-$t_i$ for path $m$,

$$\widetilde{S}_1^{(m)}(t_i;x_{65}) = n_0 - \widetilde{N}_1^{(m)}(t_i;x_{65})$$

where,

$$\widetilde{N}_1^{(m)}(t_i;x_{65}) = \sum_{i=1}^{n_0} 1_{\{\tau_i^{(m)} \leq t_i\}}$$

# HJM Simulation

- Unhedged case; the insurer's unexpected cash flow at time-t for path m is

$$ {}^{u}CF_1^{(m)}(t) = S_1(0,t;x_{65}) - \widetilde{S}_1^{(m)}(t). $$

- Survivor swap based on the Australian male population.

$$ {}^{h}CF_1^{(m)}(t) = \widehat{S}_1^{(m)}(t) - \widetilde{S}_1^{(m)}(t), $$

- Survivor swap based on the Swedish male population.

$$ {}^{h}CF_2^{(m)}(t) = S_1(0,t;x_{65}) - \widetilde{S}_1^{(m)}(t) + \varphi(\widehat{S}_2^{(m)}(t) - S_2(0,t;x_{65})), $$

# Hedge Efficiency - Australian Annuity Portfolio

- The hedge efficiency is given by: $1 - \frac{\sigma_{\text{hedged}}}{\sigma_{\text{unhedged}}}$
- Measure the reduction in unexpected cash flow.
- $\varphi$ determined from simulation.

| | Swap Population | Portfolio Size 200 | 1000 | 10,000 | Swap Ratio $\varphi$ |
|---|---|---|---|---|---|
| Dependent | Australian Index | 56.58% | 78.59% | 92.94% | 1 |
| Factor | Swedish Index | 39.67% | 53.91% | 56.67% | 1.1 |

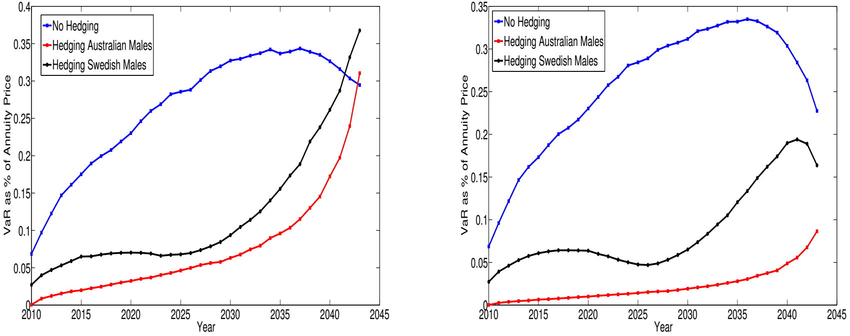
Table: Hedge Efficiency

# Solvency Capital Requirement

- Insurer remain solvent over a 1-year horizon with 99.5% probability.
- HJM model includes systematic longevity risk.
- Value future liabilities at every time-t.
- Test proposed model with different portfolio sizes.

# Solvency Capital Requirement

- Dependent factor model




(a) Portfolio Size 1,000

(b) Portfolio Size 10,000

# Conclusion

- Present a forward mortality framework for multiple populations.
- Models are fit to historical mortality data.
- Finite dimensional realisation provides a very flexible model.
- Closed-form cohort survivor curves.
- Analyse the hedge efficiency and VaR of a Survivor swap on an annuity portfolio.



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# Thank you

- Questions!

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