Forward mortality rates

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Agenda

- Why forward mortality rates?
- Defining forward mortality rates
- Market consistent measure
- Risk assessment
- Discussion
Why forward mortality rates?

- Valuing technical provisions and pricing longevity-linked securities requires consistent expectations of future mortality rates
  - C.f. forward interest rates embedded in yield curve for bond pricing
- Other approaches to forward mortality rates
  - Non-parametric – Zhu and Bauer (2011a, b, 2014)
  - Olivier-Smith model – Olivier and Jeffrey (2004), Smith (2005)
Defining forward mortality rates

- Hypothetical market in “longevity zeros” with price

\[
\text{Price}(t, \tau) = B(\tau, \tau + t) \mathbb{E}_\tau t P_{x,\tau}
\]

- Define

\[
t P_{x,\tau}(\tau) = \mathbb{E}_\tau t p_{x,\tau} = \mathbb{E}_\tau \exp \left( - \sum_{u=0}^{t-1} \mu_{x+u,\tau+u} \right)
\]

- Forward mortality rates in discrete time

\[
\nu_{x,t}(\tau) = - \ln \left( \frac{t-\tau+1 P_{x-t+\tau,\tau}(\tau)}{t-\tau P_{x-t+\tau,\tau}(\tau)} \right)
\]

\[
t P_{x,\tau}(\tau) = \exp \left( - \sum_{u=0}^{t-1} \nu_{x+u,\tau+u}(\tau) \right)
\]
Defining forward mortality rates

- We identify
  \[ \nu_{x,t}(\tau) = \mathbb{E}_\tau \mu_{x,t} \]

- Approximation due to Jensen’s inequality but tested numerically and reasonable (within 0.1%) across most ages and years

- Assume that short mortality rates are modelled by an age/period/cohort mortality model – Hunt and Blake (2014d)
  \[ \ln(\mu_{x,t}) = \eta_{x,t} = \alpha_x + \beta_x^T \kappa_t + \gamma_{t-x} \]

- Then
  \[ \nu_{x,t}(\tau) = \exp \left( \alpha_x + \beta_x^T \mathbb{E}_\tau \kappa_t + \frac{1}{2} \beta_x^T \text{Var}_\tau(\kappa_t) \beta_x + \mathbb{E}_\tau \gamma_{t-x} + \frac{1}{2} \text{Var}_\tau(\gamma_{t-x}) \right) \]
Defining forward mortality rates

• Assume random walk with drifts for the period functions
  \[ \kappa_t = \mu X_t + \kappa_{t-1} + \epsilon_t \]

• Deterministic functions may be included in drift, \( X_t \), for identifiability reasons – Hunt and Blake (2014b,c)

• Therefore

\[
\mathbb{E}_\tau \kappa_t = \kappa_\tau + \mu \sum_{s=\tau+1}^{t} X_s \\
Var_\tau(\kappa_t) = (t - \tau)\Sigma
\]
Defining forward mortality rates

- Use Bayesian approach to model and project the cohort parameters
- Fitted parameter estimates based on partial information
- Assume annual observations of each cohort providing new information
- Cohort parameter only known with certainty once observed over its entire life
Defining forward mortality rates

- Details get quite involved – see Hunt and Blake (2014a)

\[ E_{\tau \gamma_y} = M(y, \tau) = \sum_{s=0}^{\infty} \left[ \prod_{r=0}^{s-1} (1 - D_{\tau-y+r}) \right] \rho^s \left[ \gamma_{y-s}(\tau) + (1 - D_{\tau-y+s}) \beta(Y_{y-s} - \rho Y_{y-s-1}) \right] \]

\[ Var_{\tau \gamma_y} \equiv V(y, \tau) = \sum_{s=0}^{\infty} \left[ \prod_{r=0}^{s-1} (1 - D_{t-y+r})^2 \right] (1 - D_{t-y+s}) \rho^{2s} \sigma^2 \]

- However, this approach is necessary for measuring risk, as discussed later
Defining forward mortality rates

- Together, these give the forward mortality surface
- Difference $< 0.1\%$, due to rounding errors in simulations
Market consistent measure

- In order to value liabilities or value securities, we need to convert the forward mortality surface from the historic to a market consistent measure.

- Use Esscher transform, see Gerber and Shiu (1994)

\[
\mathbb{E}^Q_{exp}(\eta) = \frac{\mathbb{E}^P_{exp}(Z\eta)}{\mathbb{E}^P_{exp}(Z)}
\]

\[
Z_{x,t} = \beta_x^T \Lambda \kappa_t + \lambda^\gamma \gamma_{t-x}
\]

\[
\Lambda = \begin{pmatrix}
\lambda^{(1)} & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & \lambda^{(N)}
\end{pmatrix}
\]

\[
\nu^Q_{x,t}(\tau) = \exp (\beta_x^T \Lambda \text{Var}_T^P(\kappa_t) \beta_x + \lambda^\gamma \text{Var}_T^P(\gamma_{t-x})) \nu^P_{x,t}(\tau)
\]
Market consistent measure

- Values of market prices of longevity risk, $\lambda^{(j)}$, found from:
  - prices of traded longevity securities (if they exist) or
  - deterministic projection of mortality (e.g., CMI Projection Model)

### Period Life Expectancy at 65

![Graph showing period life expectancy at 65 across years from 2000 to 2050. The graph includes various lines for different methods such as LC, CBDX, APC, RP, GP, Observed, and CMI 1.75%.]
Market consistent measure

- Consistent prices for liabilities and securities can now be found using the same forward mortality surface.
Risk assessment

- For many purposes, we need to know how the forward mortality surface updates
  - E.g., Value at Risk, hedging

- This depends upon how the period and cohort functions update with one year’s extra observations

- NB – by tower property of conditional expectations, have

\[ \nu_{x,t}(\tau) = \mathbb{E}_\tau \nu_{x,t}(\tau + 1) \]

- Period functions are straightforward

\[ \mathbb{E}_{\tau + 1} \kappa_t = \mathbb{E}_\tau \kappa_t + \epsilon_{\tau + 1} \]

\[ \text{Var}_{\tau + 1}(\kappa_t) = (t - \tau - 1) \Sigma \]
Risk assessment

- Cohort functions, need to use Bayesian approach and assumed data generating process

\[\gamma_y(\tau + 1) = \gamma_y(\tau) + d_{\tau+1-y} \gamma_y^{\tau+1-y}\]

\[\gamma_y^{\tau+1-y}|\mathcal{F}_{\tau,y}, \beta, \rho, \sigma^2 \sim N(\beta Y_y + \rho (M(y-1, \tau) - \beta Y_{y-1}), V(y-1, \tau) + \frac{\sigma^2}{d_{\tau+1-y}})\]
Risk assessment

- Using this framework, we can update the forward mortality surface by one year and recalculate liability values or securities prices
- Value at Risk
Risk assessment

- Solvency II SCR is the 99.5% VaR of the technical provisions
- Therefore, forward rate model can calculate SCR by repeated updates of forward mortality surface
  - Avoids nested sims for SCR
- Compare with Solvency II standard model - 20% shock to mortality to proxy for VaR
  - C.f., Börger (2010)
Risk assessment

\[ \text{Risk Margin} = CoC \times \sum_{s=0}^{\infty} \text{SCR}(s)(1 + r_s)^{-s} \]

- Calculation of risk margin suffers from calculation problems
- Short rate approach:
  - Needs simulations (to give liabilities at \( s+1 \)) within simulations (to give VaR at \( s \)) within simulations (to model the run off of liabilities to \( s \))
- Forward mortality rate approach
  - Needs simulations (to give VaR at \( s \)) within simulations (to model the run off of liabilities to \( s \))
  - Progress, but not the complete answer
Risk assessment

- EIOPA (2014) suggests projecting deterministically to time $t$ to avoid nested simulations
- May distort estimation of VaR, especially in tails
- We propose alternative approach based on limited number of model points

Algorithm 1 Approximate estimation of the risk margin

1: Perform $N$ simulations to obtain empirical distribution of $\mathcal{L}(\tau + 1)$ for estimation of $\text{SCR}(\tau)$;
2: Select $p$ sets of latent variables $\{\kappa_{\tau+1}, \gamma_{\tau+1-x}\}$ corresponding to $p$ model points in the distribution of $\mathcal{L}(\tau + 1)$;
3: Perform $N$ simulations for each model point to obtain $p$ empirical distributions of $\mathcal{L}(\tau + 2)|\mathcal{L}(\tau + 1) = \mathcal{L}^{(i)}(\tau + 1)$;
4: Calculate $\text{SCR}^{(i)}(\tau + 1)$ for each model point, and $\text{SCR}(\tau + 1) = \sum_{i=1}^{p} w_i \text{SCR}^{(i)}(\tau + 1)$ where $w_i$ are a set of weights based on the relative probability of model point $i$;
5: Repeat steps 2 and 3 for each future year until the liabilities have run off;
6: Calculate the risk margin using $\text{CoC} \times \sum_{s=0}^{\infty} \text{SCR}(s)(1 + r_s)^{-s}$
• Fix $p \times N = 10,000$
  • Trade off between high $p$ (distribution at each time) and high $N$
    (robust estimate of 99.5% VaR)
• Generally, low $p$ means lower uncertainty in estimate, but biased SCR

• If $p=10$, $\text{SCR}(0) = 5.4\%$ and $\text{Risk Margin} = 4.0\%$ of best estimate of liability value.
Discussion

- Forward mortality rates provide a useful framework for many of the issues with the valuation / risk management of longevity risk
- We have introduced a discrete time forward mortality rate framework which:
  - Is consistent with models of the short mortality rate
  - Can be calibrated easily to available data
  - Can be used with a variety of individual short rate models
  - Can be extended for different processes governing period and (more difficult) cohort functions
Selected References

Questions?

- Thank you very much for your attention and your feedback
### Addendum

Market prices of risk are dimensionless and not directly comparable across models.

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<th>$\lambda^{(2)}$</th>
<th>$\lambda^{(3)}$</th>
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- We can also look at hedging strategies for the liabilities based on longevity linked securities.