Portfolio Choice with Life Annuities under Probability Distortion

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Outline

Introduction
  Motivation
  Literature Review

Model
  Model Formulation
  Theoretical Results

Numerical Results
  Investment
  Consumption
  Annuitization Strategy

Conclusion
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Conclusion
The Need for Behavioral Economics

- Policyholder’s behaviors affect insurance companies’ operations
- Behavioral Economics help understand policyholder’s behaviors
Probability Distortion

- People overweight small probabilities and underweight large probabilities

![Weighted Cumulative Probabilities](image-url)
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Literature Review

- Yarri (1965)
  - Optimal for an individual without a bequest motive to fully annuitize
- Milevsky and Young (2007)
  - Realistically incorporated mortality-contingent payout annuities (e.g. DB plan)
- Wang and Young (2012)
  - Commutable life annuities to maximize the lifetime utility
- Young and Zariphopoulou (1999)
  - Derive stochastic differential equation for a distorted probability via stochastic differential games
Goal

- Behavioral Economics
  - Probability distortion
- Portfolio Choice
  - Investment, consumption and annuitization strategy
- Continuous-time setting
- Maximize the lifetime utility
- Commutable life annuity
Market

- A riskless asset
- A risky asset
- Commutable life annuities
  - A single premium immediate annuity with a surrender option
New Probability Distortion

- Typical distortion function
  - $w(p) = \frac{p^\delta}{(p^\delta + (1-p)^\delta)^\frac{1}{\delta}}$
- Why difficult to apply?
  - Hard to derive its stochastic differential equation
- We propose a new distortion function
  - $w(p) = 1 - \frac{1}{1 - \delta \cdot \ln(1-p)}$, $\delta > 1$
Weibull Distribution

- Why Weibull distribution for stock price?
  - Explicit hazard function
- Original SDE
  \[ dX_s = \left[-X_s^{\gamma} + \gamma X_s \gamma^{-\beta} \frac{\sigma^\beta}{\beta}\right]ds + \left(2X_s^{\gamma-\beta+1} \frac{\sigma^\beta}{\beta}\right)^{\frac{1}{2}} dB_s \]
- Distorted SDE
  \[ dX_s = \left[-X_s^{\gamma} + \gamma X_s \gamma^{-\beta} \frac{\sigma^\beta}{\beta} + 2X_s^{\gamma} \left(-1 + \frac{2\delta}{1 + \delta X_s^\beta}\right)\right]ds + \left(2X_s^{\gamma-\beta+1} \frac{\sigma^\beta}{\beta}\right)^{\frac{1}{2}} dB_s \]
- \(\beta\): shape parameter
- \(\sigma\) and \(\gamma\): scale parameters
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Model

- **Wealth dynamics**
  \[
  dW_s = \left[ r(W_s - \Pi_s) - \Pi_s^\gamma + \gamma \Pi_s^{\gamma - \beta} \frac{\sigma^\beta}{\beta} + 2\Pi_s^\gamma (-1 + \frac{2\delta}{1 + \delta \Pi_s^\beta}) - C_s + A_s \right] ds + (2\Pi_s^{\gamma - \beta} + 1 \frac{\sigma^\beta}{\beta})^{\frac{1}{2}} dB_s
  \]

- **Value function**
  \[
  U(W, A) = \sup_{\pi_s, c_s} \mathbb{E} \left[ \int_0^\infty e^{-(r+\lambda)s} u(c_s) ds \right] \mid W_0 = W, A_0 = A
  \]

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Model

*HJB equation*

\[(r + \lambda)U = (rW_s + A_s)U_w + \max_{\pi_s} \left\{ \left[ -r\Pi_s - \Pi_s^\gamma + \gamma\Pi_s^\gamma - \beta \sigma^\beta \right] U_w + \Pi_s^\gamma - \beta + 1 \sigma^\beta U_{ww} \right\} + \max_c \left( \frac{c_s^{1-\gamma}}{1-\gamma} - cU_w \right) \]
Numerical Method

\[ U_w(i, j + 1) = U_w(i, j) + [W(2) - W(1)] \cdot U_{ww}(i, j + 1) \]

\[ U(i, j + 1) = U(i, j) + [W(2) - W(1)] \cdot U_w(i, j + 1) \]
Parameters

- $r = 0.04$
- $\lambda = 0.04$
- $\beta = 1$
- $\sigma = 5$
- $\gamma = 2$
- $\alpha = 2.5$
- $p = 0.2$
- $\delta = 2$
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Investment

\[-X_s \gamma + \gamma X_s^{\gamma - \beta} \frac{\sigma^\beta}{\beta} + 2X_s^{\gamma - \frac{2\delta}{1+\delta X_s^\beta}} > 0\]
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Utility Function

Before Distortion (less risk averse)

After Distortion (more risk averse)

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Portfolio Choice under Probability Distortion
## Annuitization Strategy

<table>
<thead>
<tr>
<th>Behavior</th>
<th>Utility</th>
</tr>
</thead>
<tbody>
<tr>
<td>Buy</td>
<td>$U(A+4,W-50)$</td>
</tr>
<tr>
<td>Do nothing</td>
<td>$U(A,W)$</td>
</tr>
<tr>
<td>Surrender</td>
<td>$U(A-4,W+40)$</td>
</tr>
</tbody>
</table>

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Portfolio Choice under Probability Distortion
Annuitization Strategy

Undistorted Case

+ : buy
○ : do nothing
× : surrender

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Distorted Case

+ : buy
◦ : do nothing
× : surrender
Annuitization Strategy

- Need more annuities to against fear
  - Stop buying annuity at a higher level
  - Begin surrendering annuity at a higher level

- Different $z_0$: critical ratio of wealth-to-annuity
  - An unique $z_0$ in Wang and Young (2012)

- Behavior pattern
  
  \[
  \begin{array}{c|c}
  \text{Wealth} & \text{Surrender} \\
  \hline
  \text{Annuity} & \text{Surrender} \rightarrow \text{Do nothing} \\
  & \text{Do nothing} \\
  & \text{Do nothing} \rightarrow \text{Buy} \\
  & \text{Buy} \\
  \end{array}
  \]
## Illustration

<table>
<thead>
<tr>
<th></th>
<th>No Distortion</th>
<th>Distortion</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Stock</strong></td>
<td>4.97</td>
<td>4.57</td>
</tr>
<tr>
<td><strong>Bond</strong></td>
<td>495.00</td>
<td>445.43</td>
</tr>
<tr>
<td><strong>Consumption</strong></td>
<td>138.42</td>
<td>145.36</td>
</tr>
<tr>
<td><strong>Annuitization</strong></td>
<td>Do nothing</td>
<td>Buy</td>
</tr>
</tbody>
</table>

One year later...

<table>
<thead>
<tr>
<th></th>
<th>No Distortion</th>
<th>Distortion</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>W=468 A=62</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Distortion</strong></td>
<td>W=422 A=66</td>
<td></td>
</tr>
</tbody>
</table>
Illustration

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<tr>
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</tr>
</thead>
<tbody>
<tr>
<td>Stock</td>
<td>4.96</td>
</tr>
<tr>
<td>Bond</td>
<td>995.04</td>
</tr>
<tr>
<td>Consumption</td>
<td>149.04</td>
</tr>
<tr>
<td>Annuity</td>
<td>Surrender</td>
</tr>
</tbody>
</table>

One year later...

<table>
<thead>
<tr>
<th>No Distortion</th>
<th>Distortion</th>
</tr>
</thead>
<tbody>
<tr>
<td>W=1030 A=70</td>
<td></td>
</tr>
<tr>
<td>W=998 A=74</td>
<td></td>
</tr>
</tbody>
</table>

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Portfolio Choice under Probability Distortion
Sensitivity Analysis

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Sensitivity Analysis

Annuity Buy/Surrender Behavior (delta=2)

Annuity Buy/Surrender Behavior (delta=5)

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Conclusion

- Probability distortion brings more fear

- To against fear
  - Invest less on risky asset
  - Consume more
  - Need more annuity (also support more consumption)

- Contribution of this work
  - A new distortion function
  - Weibull distribution for stock price
  - Annuitzation behavior available for each pair of (Wealth, Annuity)
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Thank you!