Minimizing Conditional Value-at-Risk under Constraint on Expected Value

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Abstract

This paper gives a complete solution to the problem of the type

$$\inf_{X \in \mathcal{F}} E[(x-X)^+]$$
$$\text{subject to} \quad E[X] \geq z, \quad \tilde{E}[X] = x_r, \quad x_d \leq X \leq x_u \ a.s.$$

where the constants satisfy $-\infty < x_d < x_r < x_u \leq \infty$, $x \in \mathbb{R}$, $z \in \mathbb{R}$. The expectations $E[\cdot]$ and $\tilde{E}[\cdot]$ are taken under two equivalent probability measures P and $\tilde{P}$ under the assumption that the Radon-Nikodým derivative has a continuous distribution. The result is then used to find the optimal dynamic portfolio which minimizes the Conditional Value-at-Risk in a complete market model.

Key Words: risk minimization, Conditional Value-at-Risk, coherent risk measure, convex risk measure, portfolio selection, portfolio optimization and hedging, convex duality, Neyman-Pearson problem

JEL Classification: G11, G32, C60

Mathematics Subject Classification (2000): 91B28, 62C25, 90C15

1 Introduction to the Main Problem

The portfolio selection problem studied in Markowitz [9] is set up as an optimization problem with the objective of maximizing expected return, subject to the constraint of variance being bounded above. Bielecki et. al. [2] solved the reverse problem in a dynamic setting with the objective of minimizing variance, subject to the expected return being bounded below. In both cases, the measure of risk of the portfolio is chosen as variance. In this work, we attempt to replace variance with a more modern choice of risk measure. Although Value-at-Risk is the most dominant risk measure used in practice, Artzner et. al. [3] and [4] have proposed general axioms for coherent risk measure, a standard measured by which Value-at-Risk has failed. Our choice is the Conditional Value-at-Risk (CVaR) which is a coherent risk measure defined incrementally based on Value-at-Risk. For references on CVaR, see Acerbi and Tasche [1], Rockafellar and Uryasev [11] and [12], and Föllmer and Schied [7]. In the following, we look for the optimal investment strategy to minimize CVaR of the final portfolio value while requiring its expectation to be above a constant in a dynamic continuous-time complete market setting.

Suppose the interest rate is a constant r and the risky asset S_t is a real-valued semimartingale process on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F})_{0\leq t\leq T}, P)$ that satisfies the usual conditions where $\mathcal{F}_0$ is trivial and $\mathcal{F}_T = \mathcal{F}$. The value of a self-financing portfolio X_t which invests ξ_t shares in the risky asset evolves according to the dynamics

$$dX_t = \xi_t dS_t + r(X_t - \xi_t S_t)dt, \quad X_0 = x_0.$$

We are looking for a strategy $(\xi_t)_{0\leq t\leq T}$ to minimize the conditional Value-at-Risk at level $0 < \lambda < 1$ of the final portfolio value: $\inf_{\xi_t} CVaR_\lambda(X_T)$, while requiring the expected value to remain above constant z: $E[X_T] \geq z$. In addition, we allow uniform bounds on the value of the portfolio over time: $x_d \leq X_t \leq x_u$ *a.s.*, $\forall t \in [0,T]$, where the constants satisfy $-\infty < x_d < x_0 < x_u \leq \infty$. Therefore, our **Main Problem** is

$$\begin{aligned}
(1) \qquad & \inf_{\xi_t} CVaR_\lambda(X_T) \\
\text{subject to} \quad & E[X_T] \geq z, \\
& x_d \leq X_t \leq x_u \ a.s., \quad \forall t \in [0,T].
\end{aligned}$$

Assumption 1.1 *Assume there is no arbitrage and the market is complete with a unique equivalent local martingale measure $\tilde{P}$ where the Radon-Nikodým derivative $\frac{d\tilde{P}}{dP}$ has a continuous distribution.*

Then any $\mathcal{F}$-measurable random variable can be replicated by a dynamic portfolio. The above dynamic optimization problem can be reduced to a static one

$$\begin{aligned}
(2) \qquad & \inf_{X\in\mathcal{F}} CVaR_\lambda(X) \\
\text{subject to} \quad & E[X] \geq z, \quad \tilde{E}[X] = x_r, \quad x_d \leq X \leq x_u \ a.s..
\end{aligned}$$

Here the expectation E is taken under the physical probability measure P, and the expectation $\tilde{E}$ is taken under the risk neutral probability measure $\tilde{P}$, while $x_r = x_0 e^{rT}$. To solve the main problem in an incomplete market setting, the exact hedging argument that translate the dynamic problem (1) into the static problem (2) has to be replaced by a super-hedging argument. This is done for expected shortfall minimization in Föllmer and Leukert [6], and for convex risk minimization in Rudloff [13]. Similarly, the hedging result can be easily adapted for S_t to be $\mathbb{R}^d$-valued, where the dimension d is a natural number. The second part of the assumption, namely the Radon-Nikodým derivative $\frac{d\tilde{P}}{dP}$ has a continuous distribution, is also made not because of technical impossibility, but because of the simplification it brings to the presentation for its lengthy discussion does not bring additional new insight to the main topic of this paper.

Using the equivalence between conditional Value-at-Risk and the Fenchel-Legendre dual of the expected shortfall derived in Rockafellar and Uryasev ([11] and [12]),

$$CVaR_\lambda(X) = \frac{1}{\lambda} \inf_{x \in \mathbb{R}} \left(E[(x - X)^+] - \lambda x \right), \quad \forall \lambda \in (0,1), \tag{3}$$

the static optimization problem can be further reduced to a two-step static optimization we name as

Two-Constraint Problem:

Step 1: Minimization of Expected Shortfall

$$\begin{aligned} & v(x) = \inf_{X \in \mathcal{F}} E[(x - X)^+] \\ \text{subject to} \quad & E[X] \geq z, \quad (\textit{return constraint}) \\ & \tilde{E}[X] = x_r, \quad (\textit{capital constraint}) \\ & x_d \leq X \leq x_u \ a.s.; \end{aligned} \tag{4}$$

Step 2: Minimization of conditional Value-at-Risk

$$\inf_{X \in \mathcal{F}} CVaR_\lambda(X) = \frac{1}{\lambda} \inf_{x \in \mathbb{R}} \left(v(x) - \lambda x \right). \tag{5}$$

Without the condition on the expectation $E[X] \geq z$, we name the problem as

One-Constraint Problem:

Step 1: Minimization of Expected Shortfall

$$
\begin{aligned}
(6) \qquad & v(x) = \inf_{X \in \mathcal{F}} E[(x-X)^+] \\
\text{subject to} \quad & \tilde{E}[X] = x_r, \quad (\textit{capital constraint}) \\
& x_d \le X \le x_u \ a.s.;
\end{aligned}
$$

Step 2: Minimization of conditional Value-at-Risk

$$
(7) \qquad \inf_{X \in \mathcal{F}} CVaR_\lambda(X) = \frac{1}{\lambda} \inf_{x \in \mathbb{R}} \left(v(x) - \lambda x \right).
$$

The solution to the problem of *Minimization of Expected Shortfall* in (6) is given in Föllmer and Leukert [6]; the solution to the problem of *Minimization of CVaR* in (7), and thus the main problem in (1) and (2) without return constraint is given in Schied [15], Sekine [16], and Li and Xu [8]. With the additional condition on the expectation $E[X] \ge z$, Rockafellar and Uryasev [11] provides a linear programming solution for the Monte-Carlo simulation of the one-time step problem. The dynamic solution given in Ruszczyński and Shapiro [14] requires the modification of the CVaR into a dynamic version. The new results obtained in this paper is to provide a solution to the problem of *Minimization of Expected Shortfall* in (4) under the condition on the expectation $E[X] \ge z$, and thus the solution to the problem of *Minimization of CVaR* in (1) and (2) under the same condition.

Föllmer and Leukert [6] derived the optimal solution to **Step 1** of the **One-Constraint Problem**,

$$
(8) \qquad X(x) = x_d \mathbb{I}_{\left\{ \frac{d\tilde{P}}{dP} > a \right\}} + x \mathbb{I}_{\left\{ \frac{d\tilde{P}}{dP} \le a \right\}}, \quad \text{for } x_d < x < x_u.
$$

The above X is the solution under a special case when the Radon-Nikodým derivative $\frac{d\tilde{P}}{dP}|_T$ is restricted to have a continuous distribution to minimize the complication in its presentation. The optimality of X can be proved in various ways, but it is clearly a result of Neyman-Pearson lemma once the connection between the problem of Minimization of Expected Shortfall and that of hypothesis testing between P and $\tilde{P}$ is established. To view it as a solution from convex duality approach, see Theorem 1.19 in Xu [17]. A direct method using Lagrange multiplier for convex optimization, a simplified version to that in the proof of Proposition 2.14, is yet another nice approach. Note that in (8), a is computed from the budget constraint $\tilde{E}[X] = x_r$ for every fixed constant x. To proceed to **Step 2**, Li and Xu [8] varied the value of x and looked for the best

x^*. Define set $A = \left\{\omega \in \Omega : \frac{d\tilde{P}}{dP}(\omega) > a\right\}$. Let a^* and A^* be the solution to the equation $\frac{1}{a} = \frac{\lambda - P(A)}{1-\tilde{P}(A)}$. Define $x^* = \frac{x_r - x_d \tilde{P}(A^*)}{1-\tilde{P}(A^*)}$. Under some technical conditions, the solution to **Step 2** of the **One-Constraint Problem** is shown in Li and Xu [8] to be

$$X^* = x_d \mathbb{I}_{A^*} + x^* \mathbb{I}_{A^{*c}}, \quad \textbf{(Two-Line Configuration)} \tag{9}$$
$$CVaR_\lambda(X^*) = -x_r + \frac{1}{\lambda}(x^* - x_d)\left(P(A^*) - \lambda \tilde{P}(A^*)\right),$$

regardless whether $x_u < \infty$ or $x_u = \infty$. More general solutions in the case when the Radon-Nikodým derivative $\frac{d\tilde{P}}{dP}|_T$ is not restricted to have a continuous distribution is presented in detail with computational examples in Li and Xu [8]. Note that the two-line configuration in (9) is inherited from the Neyman-Pearson lemma. We will see in Section 2 that when $x_u < \infty$, under some technical conditions, the solutions to both **Step 1** and **Step 2** of the **Two-Constraint Problem**, and thus the **Main Problem** (1) and (2), turn out to be a three-line configuration of the form

$$X^{**} = x_d \mathbb{I}_{A^{**}} + x^{**} \mathbb{I}_{B^{**}} + x_u \mathbb{I}_{D^{**}},, \quad \textbf{(Three-Line Configuration)}$$

where x^{**}, as well as $A^{**} = \left\{\omega \in \Omega : \frac{d\tilde{P}}{dP}(\omega) > a^{**}\right\}$, $B^{**} = \left\{\omega \in \Omega : b^{**} \le \frac{d\tilde{P}}{dP}(\omega) \le a^{**}\right\}$ and $D^{**} = \left\{\omega \in \Omega : \frac{d\tilde{P}}{dP}(\omega) < b^{**}\right\}$ are associated to the optimal choice of a^{**} and b^{**}. When $x_u = \infty$, the optimal solution X most likely will not exist, but the infimum of the CVaR can still be computed.

The key to finding the exact solution to the main problem without return constraint, is to find the pair (a^*, x^*) in the **One-Constraint Problem**; or the triplet (a^{**}, b^{**}, x^{**}) in the **Two-Constraint Problem**. In the first case, Theorem 2.10 and Remark 2.11 in Li and Xu [8] state that (a^*, x^*) is the solution to the *capital constraint* ($\tilde{E}[X] = x_r$) and *first order Euler condition* ($v'(x) = 0$ in **Step 2**):

$$x_d \tilde{P}(A) + x\tilde{P}(A^c) = x_r,$$
$$P(A) + \frac{\tilde{P}(A^c)}{a} - \lambda = 0.$$

In the second case, we will see in Proposition 2.14 and Theorem 2.15 that (a^{**}, b^{**}, x^{**}) is the solution to the same two conditions plus the *return constraint* ($E[X] = z$):

$$x_d P(A) + x P(B) + x_u P(D) = z,$$
$$x_d \tilde{P}(A) + x \tilde{P}(B) + x_u \tilde{P}(D) = x_r,$$
$$P(A) + \frac{\tilde{P}(B) - bP(B)}{a - b} - \lambda = 0.$$

The main theorems about the solutions to both **Step 1** and **Step 2** to the **Two-Constraint Problem** are stated in Section 2; their proofs are recorded in Section 3; Section 4 lists possible future work.

2 Solution to the Main Problem

2.1 Case: $x_u < \infty$

Before establishing their existence, we first define some particular Two-Line Configurations and the general Three-Line Configuration that satisfy their respective capital and expected return constraints. Recall the definitions of the sets

$$A = \left\{\omega \in \Omega \,:\, \tfrac{d\tilde{P}}{dP}(\omega) > a\right\}, \quad B = \left\{\omega \in \Omega \,:\, b \leq \tfrac{d\tilde{P}}{dP}(\omega) \leq a\right\}, \quad D = \left\{\omega \in \Omega \,:\, \tfrac{d\tilde{P}}{dP}(\omega) < b\right\}. \tag{10}$$

Definition 2.1 ***Three-Line Configuration*** *has the structure* $X = x_d \mathbb{I}_A + x \mathbb{I}_B + x_u \mathbb{I}_D$.

Two-Line Configuration $X = x\mathbb{I}_B + x_u \mathbb{I}_D$ *is always associated to the definitions* $a = \infty$, $B = \left\{\omega \in \Omega \,:\, \frac{d\tilde{P}}{dP}(\omega) \geq b\right\}$ *and* $D = \left\{\omega \in \Omega \,:\, \frac{d\tilde{P}}{dP}(\omega) < b\right\}$.

Two-Line Configuration $X = x_d\mathbb{I}_A + x \mathbb{I}_B$ *is always associated to the definitions* $b = 0$, $A = \left\{\omega \in \Omega \,:\, \frac{d\tilde{P}}{dP}(\omega) > a\right\}$, *and* $B = \left\{\omega \in \Omega \,:\, \frac{d\tilde{P}}{dP}(\omega) \leq a\right\}$.

Two-Line Configuration $X = x_d\mathbb{I}_A + x_u \mathbb{I}_D$ *is always associated to the definitions* $a = b$, $A = \left\{\omega \in \Omega \,:\, \frac{d\tilde{P}}{dP}(\omega) > a\right\}$, *and* $D = \left\{\omega \in \Omega \,:\, \frac{d\tilde{P}}{dP}(\omega) < a\right\}$.

General Constraints *are the capital constraint and the equality part of the expected return constraint for* ***Three-Line Configuration*** $X = x_d \mathbb{I}_A + x \mathbb{I}_B + x_u \mathbb{I}_D$*:*

$$E[X] = x_d P(A) + x P(B) + x_u P(D) = z,$$
$$\tilde{E}[X] = x_d \tilde{P}(A) + x \tilde{P}(B) + x_u \tilde{P}(D) = x_r.$$

Degenerated Constraints 1 *are the capital constraint and the equality part of the expected return constraint for* ***Two-Line Configuration*** $X = x\mathbb{I}_B + x_u\mathbb{I}_D$*:*

$$E[X] = xP(B) + x_uP(D) = z,$$
$$\tilde{E}[X] = x\tilde{P}(B) + x_u\tilde{P}(D) = x_r.$$

Degenerated Constraints 2 *are the capital constraint and the equality part of the expected return constraint for* ***Two-Line Configuration*** $X = x_d\mathbb{I}_A + x\mathbb{I}_B$*:*

$$E[X] = x_dP(A) + xP(B) = z,$$
$$\tilde{E}[X] = x_d\tilde{P}(A) + x\tilde{P}(B) = x_r.$$

Degenerated Constraints 3 *are the capital constraint and the equality part of the expected return constraint for* ***Two-Line Configuration*** $X = x_d\mathbb{I}_A + x_u\mathbb{I}_D$*:*

$$E[X] = x_dP(A) + x_uP(D) = z,$$
$$\tilde{E}[X] = x_d\tilde{P}(A) + x_u\tilde{P}(D) = x_r.$$

Note that **Degenerated Constraints 1** corresponds to the **General Constraints** when $a = \infty$; **Degenerated Constraints 2** corresponds to the **General Constraints** when $b = 0$; and **Degenerated Constraints 3** corresponds to the **General Constraints** when $a = b$.

Definition 2.2 *For fixed* $-\infty < x_d < x_r < x_u < \infty$*, let* $\bar{a} = \bar{b}$ *be the constant that satisfies capital constraint* $\tilde{E}[X] = x_d\tilde{P}(A) + x_u\tilde{P}(D) = x_r$ *for configuration* $X = x_d\mathbb{I}_A + x_u\mathbb{I}_D$ *in* ***Degenerated Constraints 3****. Consequently,* $\bar{A}$*,* $\bar{D}$ *and* $\bar{X}$ *are associated to the constant* $\bar{a} = \bar{b}$*, i.e.,* $\bar{X} = x_d\mathbb{I}_{\bar{A}} + x_u\mathbb{I}_{\bar{D}}$ *where* $\bar{A} = \left\{\omega \in \Omega : \frac{d\tilde{P}}{dP}(\omega) > \bar{a}\right\}$*, and* $\bar{D} = \left\{\omega \in \Omega : \frac{d\tilde{P}}{dP}(\omega) < \bar{a}\right\}$*. Define* $\bar{z} = E[\bar{X}] = x_dP(\bar{A}) + x_uP(\bar{D})$*.*

Note that $\bar{z}$ is the unique expected value of a Two-Line configuration that satisfy **Degenerated Constraints 3**.

Lemma 2.3 $\bar{z}$ *is the highest return that can be obtained by a portfolio with initial capital* x_0 *and is bounded between* x_d *and* x_u*:*

$$\bar{z} = \max_{X \in \mathcal{F}} E[X] \quad s.t. \quad \tilde{E}[X] = x_r, \quad x_d \leq X \leq x_u \ a.s..$$

From now on, we will concern ourselves with $z \in [x_r, \bar{z}]$. The lower bound can be interpreted that the investment will yield a higher return than the risk-free rate r, i.e., $z = E[X] \geq x_0 e^{rT} = x_r$. Mathematically, when $z \in (-\infty, x_r)$, the optimal solution X^* to the **One-Constraint Problem** satisfies the *return constraint* $E[X^*] \geq z$ automatically (see Lemma 2.4), thus it is also the optimal solution to the **Two-Constraint Problem**. We refer to Li and Xu [8] for the details. For convenience, we will state in Theorem 2.11 the main results from Li and Xu [8] about X^* under additional Assumption 1.1.

Lemma 2.4 *For fixed $-\infty < x_d < x_r < x_u < \infty$, and any $x \in [x_d, x_r]$, choose b so that configuration $X = x\mathbb{I}_B + x_u\mathbb{I}_D$ satisfies the capital constraint $\tilde{E}[X] = x\tilde{P}(B) + x_u\tilde{P}(D) = x_r$ in **Degenerated Constraints 1**. Let $z = E[X] = xP(B) + x_uP(D)$. Then z decreases continuously from $\bar{z}$ to x_r as x increases from x_d to x_r. For any $x \in [x_r, x_u]$, choose a so that configuration $X = x_d\mathbb{I}_A + x\mathbb{I}_B$ satisfies the capital constraint $\tilde{E}[X] = x_d\tilde{P}(A) + x\tilde{P}(B) = x_r$ in **Degenerated Constraints 2**. Let $z = E[X] = x_dP(A) + xP(B)$. Then z increases continuously from x_r to $\bar{z}$ as x increases from x_r to x_u.*

From the above lemma, we can see that for given x value, we can compute the corresponding z value in **Degenerated Constraints 1** and **Degenerated Constraints 2**. Since their relationship is monotone and continuous in each situation, given z we can find the corresponding x value in both situations.

Definition 2.5 *For fixed $-\infty < x_d < x_r < x_u < \infty$, and fixed $z \in [x_r, \bar{z}]$, define x_{z1} and x_{z2} to be the corresponding x values for configurations that satisfy **Degenerated Constraints 1** and **Degenerated Constraints 2** respectively.*

Definition 2.5 means that when we fix z in a proper interval $[x_r, \bar{z}]$, we can find two feasible solutions: $X = x_{z1}\mathbb{I}_B + x_u\mathbb{I}_D$ satisfying $\tilde{E}[X] = x_{z1}\tilde{P}(B) + x_u\tilde{P}(D) = x_r$ and $E[X] = x_{z1}P(B) + x_uP(D) = z$; $X = x_d\mathbb{I}_A + x_{z2}\mathbb{I}_B$ satisfying $\tilde{E}[X] = x_d\tilde{P}(A) + x_{z2}\tilde{P}(B) = x_r$ and $E[X] = x_dP(A) + x_{z2}P(B) = z$.

Now if we fix $x \in [x_d, x_{z1}]$, and as in Lemma 2.4, choose b so that configuration $X = x\mathbb{I}_B + x_u\mathbb{I}_D$ satisfies the capital constraint $\tilde{E}[X] = x\tilde{P}(B) + x_u\tilde{P}(D) = x_r$ in **Degenerated Constraints 1**. At the left end point $x = x_d$, we encounter $\bar{X}$ given by **Degenerated Constraints 3** in Definition 2.2 and corresponding $\bar{z} = E[\bar{X}] \geq z$. At the right end point $x = x_{z1}$, we encounter $X = x_{z1}\mathbb{I}_B + x_u\mathbb{I}_D$ such that $E[X] = z$. In between, $E[X]$, where $X = x\mathbb{I}_B + x_u\mathbb{I}_D$ and $\tilde{E}[X] = x_r$, is decreasing according to Lemma 2.4. We recognize that $E[X] = xP(B) + x_uP(D) \geq z$, for all $x \in [x_d, x_{z1}]$. Similar analysis can be applied to the interval $x \in [x_{z2}, x_u]$. We make this conclusion in the following lemma.

Lemma 2.6 *For fixed $-\infty < x_d < x_r < x_u < \infty$, and fixed $z \in [x_r, \bar{z}]$,*

1. *If we fix $x \in [x_d, x_{z1}]$, the Two-Line Configuration $X = x\mathbb{I}_B + x_u\mathbb{I}_D$ which satisfies the capital constraint $\tilde{E}[X] = x\tilde{P}(B) + x_u\tilde{P}(D) = x_r$ in Degenerated Constraints 1* ***satisfies*** *the expected return constraint: $E[X] = xP(B) + x_uP(D) \geq z$;*

2. *If we fix $x \in (x_{z1}, x_r]$, the Two-Line Configuration $X = x\mathbb{I}_B + x_u\mathbb{I}_D$ which satisfies the capital constraint $\tilde{E}[X] = x\tilde{P}(B) + x_u\tilde{P}(D) = x_r$ in Degenerated Constraints 1* ***fails*** *the expected return constraint: $E[X] = xP(B) + x_uP(D) < z$;*

3. *If we fix $x \in [x_r, x_{z2})$, the Two-Line Configuration $X = x_d\mathbb{I}_A + x\mathbb{I}_B$ which satisfies the capital constraint $\tilde{E}[X] = x_d\tilde{P}(A) + x\tilde{P}(B) = x_r$ in Degenerated Constraints 2* ***fails*** *the expected return constraint: $E[X] = xP(B) + x_uP(D) < z$;*

4. *If we fix $x \in [x_{z2}, x_u]$, the Two-Line Configuration $X = x_d\mathbb{I}_A + x\mathbb{I}_B$ which satisfies the capital constraint $\tilde{E}[X] = x_d\tilde{P}(A) + x\tilde{P}(B) = x_r$ in Degenerated Constraints 2* ***satisfies*** *the expected return constraint: $E[X] = xP(B) + x_uP(D) \geq z$.*

Proposition 2.7 *For fixed $-\infty < x_d < x_r < x_u < \infty$, and fixed $z \in [x_r, \bar{z}]$, if we fix $x \in [x_d, x_{z1}]$, then there exists a* ***Two-Line Configuration*** *$X = x\mathbb{I}_B + x_u\mathbb{I}_D$ which is the optimal solution to* ***Step 1*** *of the* ***Two-Constraint Problem****; if we fix $x \in [x_{z2}, x_u]$, then there exists a* ***Two-Line Configuration*** *$X = x_d\mathbb{I}_A + x\mathbb{I}_B$ which is the optimal solution to* ***Step 1*** *of the* ***Two-Constraint Problem****.*

When $x \in (x_{z1}, x_{z2})$, the Two-Line Configurations that can be achieved with the right amount of initial capital ($\tilde{E}[X] = x_r$) do not generate high enough expected return ($E[X] < z$) to be feasible, so we have to look for a novel solution of Three-Line Configuration that is both feasible and optimal.

Lemma 2.8 *For fixed $-\infty < x_d < x_r < x_u < \infty$, fixed $z \in [x_r, \bar{z}]$, and fixed $x \in (x_{z1}, x_{z2})$, choose the pair of real numbers $-\infty < b \leq a < \infty$ so that configuration $X = x_d\mathbb{I}_A + x\mathbb{I}_B + x_u\mathbb{I}_D$ always satisfies the capital constraint $\tilde{E}[X] = x_d\tilde{P}(A) + x\tilde{P}(B) + x_u\tilde{P}(D) = x_r$ in* ***General Constraints****. When $b = \bar{b} = \bar{a} = a$, $X = \bar{X}$ and $E[\bar{X}] = \bar{z}$. When $b < \bar{b}$ and $a > \bar{a}$, the expected value $E[X] = x_dP(A) + xP(B) + x_uP(D)$ decreases continuously as b decreases and a increases. In the extreme case $b = 0$, the Three-Line configuration becomes the Two-Line Configuration $X = x\mathbb{I}_B + x_u\mathbb{I}_D$; in the extreme $a = \infty$, the Three-Line configuration becomes the Two-Line Configuration $X = x_d\mathbb{I}_A + x\mathbb{I}_B$. In either extreme cases, the expected value is below z by Lemma 2.6.*

Proposition 2.9 *For fixed $-\infty < x_d < x_r < x_u < \infty$, and fixed $z \in [x_r, \bar{z}]$, if we fix $x \in (x_{z1}, x_{z2})$, then there exists a* ***Three-Line Configuration*** *$X = x_d \mathbb{I}_A + x \mathbb{I}_B + x_u \mathbb{I}_D$ that satisfies the* ***General Constraints*** *which is the optimal solution to* ***Step 1*** *of the* ***Two-Constraint Problem****.*

Let us recall the first step for the **Two-Constraint Problem**:

Step 1: Minimization of Expected Shortfall

$$
\begin{aligned}
& v(x) = \inf_{X \in \mathcal{F}} E[(x - X)^+] \\
\text{subject to} \quad & E[X] \geq z, \quad (\textit{return constraint}) \\
& \tilde{E}[X] = x_r, \quad (\textit{capital constraint}) \\
& x_d \leq X \leq x_u \ \textit{a.s.};
\end{aligned}
$$

Theorem 2.10 (Solution to Step 1: Minimization of Expected Shortfall) *For fixed $-\infty < x_d < x_r < x_u < \infty$, and fixed $z \in [x_r, \bar{z}]$. The optimal $X(x)$ and the corresponding value function $v(x)$ to* ***Step 1: Minimization of Expected Shortfall*** *of the* ***Two-Constraint Problem*** *are as follows:*

- $x \in (-\infty, x_d]$:

$$
\begin{aligned}
X(x) = & \ \textit{any random variable with values in } [x_d, x_u] \textit{ satisfying both } \tilde{E}[X(x)] = x_r \textit{ and } E[X(x)] \geq z, \\
v(x) = & \ 0.
\end{aligned}
$$

- $x \in [x_d, x_{z1}]$:

$$
\begin{aligned}
X(x) = & \ \textit{any random variable with values in } [x, x_u] \textit{ satisfying both } \tilde{E}[X(x)] = x_r \textit{ and } E[X(x)] \geq z, \\
v(x) = & \ 0.
\end{aligned}
$$

- $x \in (x_{z1}, x_{z2})$:

$$
\begin{aligned}
X(x) = & \ x_d \mathbb{I}_{A_x} + x \mathbb{I}_{B_x} + x_u \mathbb{I}_{D_x} \textit{ where } A_x, B_x, D_x \textit{ are determined by } a_x \textit{ and } b_x \textit{ through definitions (10)} \\
& \quad \textit{satisfying the General Constraints: } \tilde{E}[X(x)] = x_r \textit{ and } E[X(x)] = z, \\
v(x) = & \ (x - x_d) P(A_x).
\end{aligned}
$$

- $x \in [x_{z2}, x_u]$:

$$X(x) = x_d \mathbb{I}_{A_x} + x \mathbb{I}_{B_x} \text{ where } A_x, B_x \text{ are determined by } a_x \text{ as in Definition 2.1 satisfying both}$$
$$\tilde{E}[X(x)] = x_r \text{ and } E[X(x)] \geq z,$$
$$v(x) = (x - x_d)P(A_x).$$

- $x \in [x_u, \infty)$:

$$X(x) = x_d \mathbb{I}_{\bar{A}} + x_u \mathbb{I}_{\bar{B}} = \bar{X} \text{ where } \bar{A}, \bar{B} \text{ are associated to } \bar{a} \text{ as in Definition 2.2 satisfying both}$$
$$\tilde{E}[X(x)] = x_r \text{ and } E[X(x)] = \bar{z} \geq z,$$
$$v(x) = (x - x_d)P(\bar{A}) + (x - x_u)P(\bar{B}).$$

To solve **Step 2** of the **Two-Constraint Problem**, we need to find

$$\frac{1}{\lambda} \inf_{x \in \mathbb{R}} (v(x) - \lambda x),$$

where we have already computed $v(x)$ in Theorem 2.10. It turns out that depending on the z level in the return constraint of the **Two-Constraint Problem**, sometimes the optimal is obtained by the Two-Line solution to the **One-Constraint Problem**, other times it is obtained by a true Three-Line solution. To accomplish this, we have to recall the results of Theorem 2.10 and Remark 2.11 in Li and Xu [8].

Theorem 2.11 (Theorem 2.10 and Remark 2.11 in Li and Xu [8] when $x_u < \infty$)

1. *Suppose* $\operatorname{ess\,sup} \frac{d\tilde{P}}{dP} \leq \frac{1}{\lambda}$. *$X = x_r$ is the optimal solution to **Step 2: Minimization of Conditional Value-at-Risk** of the **One-Constraint Problem** and the associated minimal risk is*

$$CVaR(X) = -x_r.$$

2. *Suppose* $\operatorname{ess\,sup} \frac{d\tilde{P}}{dP} > \frac{1}{\lambda}$.

   - *If $\frac{1}{\bar{a}} \leq \frac{\lambda - P(\bar{A})}{1 - \tilde{P}(\bar{A})}$ (see Definition 2.2), then $\bar{X} = x_d \mathbb{I}_{\bar{A}} + x_u \mathbb{I}_{\bar{D}}$ is the optimal solution to **Step 2: Minimization of Conditional Value-at-Risk** of the **One-Constraint Problem** and the* *associated minimal risk is*

$$CVaR(\bar{X}) = -x_r + \frac{1}{\lambda}(x_u - x_d)(P(\bar{A}) - \lambda\tilde{P}(\bar{A})).$$

- *Otherwise, let* a^* *be the solution to the equation* $\frac{1}{a} = \frac{\lambda - P(A)}{1-\tilde{P}(A)}$. *Associate sets* $A^* = \left\{\omega \in \Omega : \frac{d\tilde{P}}{dP}(\omega) > a^*\right\}$ *and* $B^* = \left\{\omega \in \Omega : \frac{d\tilde{P}}{dP}(\omega) \leq a^*\right\}$ *to level* a^*. *Define* $x^* = \frac{x_r - x_d\tilde{P}(A^*)}{1-\tilde{P}(A^*)}$ *so that configuration*

$$X^* = x_d\mathbb{I}_{A^*} + x^*\mathbb{I}_{B^*}$$

satisfies the capital constraint $\tilde{E}[X^*] = x_d\tilde{P}(A^*) + x^*\tilde{P}(B^*) = x_r$ *in* ***Degenerated Constraints 2****. Then* X^* *is the optimal solution to* ***Step 2: Minimization of Conditional Value-at-Risk*** *of the* ***One-Constraint Problem*** *and the associated minimal risk is*

$$CVaR(X^*) = -x_r + \frac{1}{\lambda}(x^* - x_d)(P(A^*) - \lambda\tilde{P}(A^*)).$$

Definition 2.12 *In part 2 of Theorem 2.11, define* $z^* = \bar{z}$ *in the first case when* $\frac{1}{\bar{a}} \leq \frac{\lambda - P(\bar{A})}{1-\tilde{P}(\bar{A})}$*; define* $z^* = E[X^*]$ *in the second case when* $\frac{1}{\bar{a}} > \frac{\lambda - P(\bar{A})}{1-\tilde{P}(\bar{A})}$.

It is straightforward to see that when z is smaller than z^*, the Two-Line solution provided in Theorem 2.11 is indeed the optimal solution to **Step 2: Minimization of Conditional Value-at-Risk** of the **Two-Constraint Problem**. While when z is greater than z^* the Two-Line solutions are no longer feasible in the **Two-Constraint Problem** and we will show now that the Three-Line solutions are not only feasible but also optimal.

For $z \in (z^*, \bar{z}]$, **Step 2** of the **Two-Constraint Problem**

$$\frac{1}{\lambda}\inf_{x\in\mathbb{R}}(v(x) - \lambda x)$$

is the minimum of the following five sub-problems after applying Theorem 2.10:

Case 1

$$\frac{1}{\lambda}\inf_{(-\infty,x_d]}(v(x) - \lambda x) = \frac{1}{\lambda}\inf_{(-\infty,x_d]}(-\lambda x) = -x_d;$$

Case 2

$$\frac{1}{\lambda}\inf_{[x_d,x_{z1}]}(v(x) - \lambda x) = \frac{1}{\lambda}\inf_{[x_d,x_{z1}]}(-\lambda x) = -x_{z1} \leq -x_d;$$

Case 3

$$\frac{1}{\lambda}\inf_{(x_{z1},x_{z2})}(v(x)-\lambda x)=\frac{1}{\lambda}\inf_{(x_{z1},x_{z2})}\left((x-x_d)P(A_x)-\lambda x\right);$$

Case 4

$$\frac{1}{\lambda}\inf_{[x_{z2},x_u]}(v(x)-\lambda x)=\frac{1}{\lambda}\inf_{[x_{z2},x_u]}\left((x-x_d)P(A_x)-\lambda x\right);$$

Case 5

$$\frac{1}{\lambda}\inf_{[x_u,\infty)}(v(x)-\lambda x)=\frac{1}{\lambda}\inf_{[x_u,\infty)}\big((x-x_d)P(\bar{A})+(x-x_u)P(\bar{B})-\lambda x\big).$$

We first establish the convexity of the objective function and its continuity in Lemma 2.13, then we prove the Three-Line solution which is feasible and satisfies the first order condition is indeed optimal.

Lemma 2.13 *$v(x)$ is a convex function for $x\in\mathbb{R}$, and thus continuous.*

Proposition 2.14 *For fixed $-\infty<x_d<x_r<x_u<\infty$, and fixed $z\in(z^*,\bar{z}]$. Suppose $\operatorname{ess\,sup}\frac{d\tilde{P}}{dP}>\frac{1}{\lambda}$. The solution a^{**},b^{**} and x^{**} (and consequently, A^{**},B^{**} and D^{**}) to the system*

$$\begin{aligned}
&x_dP(A)+xP(B)+x_uP(D)=z, \quad \textit{(return constraint)}\\
&x_d\tilde{P}(A)+x\tilde{P}(B)+x_u\tilde{P}(D)=x_r, \quad \textit{(capital constraint)}\\
&P(A)+\frac{\tilde{P}(B)-bP(B)}{a-b}-\lambda=0, \quad \textit{(first order Euler condition)}
\end{aligned}$$

*exists. $X^{**}=x_d\mathbb{I}_{A^{**}}+x^{**}\mathbb{I}_{B^{**}}+x_u\mathbb{I}_{D^{**}}$ is the optimal solution to **Step 2: Minimization of Conditional Value-at-Risk** of the **Two-Constraint Problem** where*

$$\frac{1}{\lambda}\inf_{x\in\mathbb{R}}(v(x)-\lambda x)=\frac{1}{\lambda}\min_{(x_{z1},x_{z2})}(v(x)-\lambda x),$$

and the associated minimal risk is

$$CVaR(X^{**})=\frac{1}{\lambda}\left((x^{**}-x_d)P(A^{**})-\lambda x^{**}\right).$$

Theorem 2.15 (Solution to Step 2: Minimization of Conditional Value-at-Risk) *For fixed $-\infty<x_d<x_r<x_u<\infty$.*

1. *Suppose $\operatorname{ess\,sup}\frac{d\tilde{P}}{dP}\leq\frac{1}{\lambda}$ and $z=x_r$. $X=x_r$ is the optimal solution to **Step 2: Minimization of***

Conditional Value-at-Risk *of the* ***Two-Constraint Problem*** *and the associated minimal risk is*

$$CVaR(X) = -x_r.$$

2. *Suppose* $\operatorname{ess\,sup} \frac{d\tilde{P}}{dP} \leq \frac{1}{\lambda}$ *and* $z \in (x_r, \bar{z}]$. *The optimal solution to* ***Step 2: Minimization of Conditional Value-at-Risk*** *of the* ***Two-Constraint Problem*** *does not exist and the minimal risk is*

$$CVaR(X) = -x_r.$$

3. *Suppose* $\operatorname{ess\,sup} \frac{d\tilde{P}}{dP} > \frac{1}{\lambda}$ *and* $z \in [x_r, z^*]$.

   - *If* $\frac{1}{\bar{a}} \leq \frac{\lambda - P(\bar{A})}{1-\tilde{P}(\bar{A})}$ *(see Definition 2.2), then* $\bar{X} = x_d \mathbb{I}_{\bar{A}} + x_u \mathbb{I}_{\bar{D}}$ *is the optimal solution to* ***Step 2: Minimization of Conditional Value-at-Risk*** *of the* ***Two-Constraint Problem*** *and the associated minimal risk is*

$$CVaR(\bar{X}) = -x_r + \frac{1}{\lambda}(x_u - x_d)(P(\bar{A}) - \lambda \tilde{P}(\bar{A})).$$

   - *Otherwise,* $X^* = x_d \mathbb{I}_{A^*} + x^* \mathbb{I}_{B^*}$ *defined in Theorem 2.11 is the optimal solution to* ***Step 2: Minimization of Conditional Value-at-Risk*** *of the* ***Two-Constraint Problem*** *and the associated minimal risk is*

$$CVaR(X^*) = -x_r + \frac{1}{\lambda}(x^* - x_d)(P(A^*) - \lambda \tilde{P}(A^*)).$$

4. *Suppose* $\operatorname{ess\,sup} \frac{d\tilde{P}}{dP} > \frac{1}{\lambda}$ *and* $z \in (z^*, \bar{z}]$. $X^{**} = x_d \mathbb{I}_{A^{**}} + x^{**} \mathbb{I}_{B^{**}} + x_u \mathbb{I}_{D^{**}}$ *defined in Proposition 2.14 is the optimal solution to* ***Step 2: Minimization of Conditional Value-at-Risk*** *of the* ***Two-Constraint Problem*** *and the associated minimal risk is*

$$CVaR(X^{**}) = \frac{1}{\lambda}\left((x^{**} - x_d)P(A^{**}) - \lambda x^{**}\right).$$

2.2 Case: $x_u = \infty$

We first restate Theorem 2.11 in the current context. When $x_u = \infty$, we interpret $\bar{A} = \Omega$ and $\bar{z} = \infty$.

Theorem 2.16 (Theorem 2.10 and Remark 2.11 in Li and Xu [8] when $x_u = \infty$)

1. *Suppose* $\operatorname{ess\,sup} \frac{d\tilde{P}}{dP} \leq \frac{1}{\lambda}$. $X = x_r$ *is the optimal solution to* ***Step 2: Minimization of Conditional***

Value-at-Risk of the *One-Constraint Problem* and the associated minimal risk is

$$CVaR(X) = -x_r.$$

2. *Suppose* $\operatorname{ess\,sup} \frac{d\tilde{P}}{dP} > \frac{1}{\lambda}$. *Let* a^* *be the solution to the equation* $\frac{1}{a} = \frac{\lambda - P(A)}{1-\tilde{P}(A)}$. *Associate sets* $A^* = \left\{\omega \in \Omega : \frac{d\tilde{P}}{dP}(\omega) > a^*\right\}$ *and* $B^* = \left\{\omega \in \Omega : \frac{d\tilde{P}}{dP}(\omega) \le a^*\right\}$ *to level* a^*. *Define* $x^* = \frac{x_r - x_d\tilde{P}(A^*)}{1-\tilde{P}(A^*)}$ *so that configuration*

$$X^* = x_d\mathbb{I}_{A^*} + x^*\mathbb{I}_{B^*}$$

satisfies the capital constraint $\tilde{E}[X^*] = x_d\tilde{P}(A^*) + x^*\tilde{P}(B^*) = x_r$ *in* ***Degenerated Constraints 2****. Then* X^* *is the optimal solution to* ***Step 2: Minimization of Conditional Value-at-Risk*** *of the* ***One-Constraint Problem*** *and the associated minimal risk is*

$$CVaR(X^*) = -x_r + \frac{1}{\lambda}(x^* - x_d)(P(A^*) - \lambda\tilde{P}(A^*)).$$

Theorem 2.17 (Minimization of Conditional Value-at-Risk When $x_u = \infty$) *For fixed* $-\infty < x_d < x_r < x_u = \infty$.

1. *Suppose* $\operatorname{ess\,sup} \frac{d\tilde{P}}{dP} \le \frac{1}{\lambda}$ *and* $z = x_r$. $X = x_r$ *is the optimal solution to* ***Step 2: Minimization of Conditional Value-at-Risk*** *of the* ***Two-Constraint Problem*** *and the associated minimal risk is*

$$CVaR(X) = -x_r.$$

2. *Suppose* $\operatorname{ess\,sup} \frac{d\tilde{P}}{dP} \le \frac{1}{\lambda}$ *and* $z \in (x_r, \infty)$. *The optimal solution to* ***Step 2: Minimization of Conditional Value-at-Risk*** *of the* ***Two-Constraint Problem*** *does not exist and the minimal risk is*

$$CVaR(X) = -x_r.$$

3. *Suppose* $\operatorname{ess\,sup} \frac{d\tilde{P}}{dP} > \frac{1}{\lambda}$ *and* $z \in [x_r, z^*]$. X^* *is the optimal solution to* ***Step 2: Minimization of Conditional Value-at-Risk*** *of the* ***Two-Constraint Problem*** *and the associated minimal risk is*

$$CVaR(X^*) = -x_r + \frac{1}{\lambda}(x^* - x_d)(P(A^*) - \lambda\tilde{P}(A^*)).$$

4. *Suppose* $\operatorname{ess\,sup} \frac{d\tilde{P}}{dP} > \frac{1}{\lambda}$ *and* $z \in (z^*, \infty)$. *The optimal solution to* ***Step 2: Minimization of Conditional Value-at-Risk*** *of the* ***Two-Constraint Problem*** *does not exist and the minimal risk is*

$$CVaR(X^*) = -x_r + \frac{1}{\lambda}(x^* - x_d)(P(A^*) - \lambda\tilde{P}(A^*)).$$

Example 2.18 (CVaR Minimization under Black-Scholes' Model) *Suppose an agent is trading between a money market account with interest rate* $r = 5\%$ *and one stock that follows geometric Brownian motion* $dS_t = \mu S_t dt + \sigma S_t dW_t$ *with parameter values* $\mu = 0.2$, $\sigma = 0.1$ *and* $S_0 = 10$. *The endowment starts at* $X_0 = 10$ *and bankruptcy is not allowed at any time, thus* $X_t \geq 0$ *for all* t. *The expected terminal value* $E[X_T]$ *at time horizon* $T = 2$ *is required to be above a fixed level* z. *Take* $z \in [z^*, \bar{z}]$, *where* z^* *is the optimal expected terminal value achieved by the 'one-star-system' when there is no return requirement and* $\bar{z}$ *is the highest expected value achievable. We let the final value to be bounded above by* $x_u < \infty$ *since it is of interest to see cases where the optimal of the two-constraint problem is achieved by 'double-star-system'. Recall that the triplet* (a^{**}, b^{**}, x^{**}) *in the 'double-star-system' is the solution to the system*

$$\begin{aligned} &x_d P(A) + x P(B) + x_u P(D) = z, \quad (\textit{return constraint}) \\ &x_d \tilde{P}(A) + x \tilde{P}(B) + x_u \tilde{P}(D) = x_r, \quad (\textit{capital constraint}) \\ &P(A) + \frac{\tilde{P}(B) - bP(B)}{a - b} - \lambda = 0, \quad (\textit{first order Euler condition}) \end{aligned}$$

where in the Black-Sholes model, $P(A)$, $P(B)$, $P(D)$ *and* $\tilde{P}(A)$, $tildeP(B)$, $\tilde{P}(D)$ *can be written as*

$$\begin{aligned} &P(A) = N(-\tfrac{\theta\sqrt{T}}{2} - \tfrac{\ln a}{\theta\sqrt{T}}), \quad P(D) = 1 - N(-\tfrac{\theta\sqrt{T}}{2} - \tfrac{\ln b}{\theta\sqrt{T}}), \quad P(B) = 1 - P(A) - P(D), \\ &\tilde{P}(A) = N(\tfrac{\theta\sqrt{T}}{2} - \tfrac{\ln a}{\theta\sqrt{T}}), \quad \tilde{P}(D) = 1 - N(\tfrac{\theta\sqrt{T}}{2} - \tfrac{\ln b}{\theta\sqrt{T}}), \quad \tilde{P}(B) = 1 - \tilde{P}(A) - \tilde{P}(D). \end{aligned}$$

Then the optimal portfolio, its hedging strategy and the associated minimum CVaR can be calculated as below:

$$\begin{aligned} CVaR(X^{**}) &= \frac{1}{\lambda}\left((x^{**} - x_d)P(A^{**}) - \lambda x^{**}\right) \\ X_t^{**} &= e^{-r(T-t)}[x^{**}N(d_+(a^{**}, S_t, t)) + x_d N(d_-(a^{**}, S_t, t))] \\ &\quad + e^{-r(T-t)}[x^{**}N(d_-(b^{**}, S_t, t)) + x_u N(d_+(b^{**}, S_t, t))] - e^{r(T-t)}x^{**}, \\ \xi_t^{**} &= \frac{x^{**} - x_d}{\sigma S_t\sqrt{2\pi(T-t)}} e^{-r(T-t) - \frac{d_-^2(a^{**}, S_t, t)}{2}} + \frac{x^{**} - x_u}{\sigma S_t\sqrt{2\pi(T-t)}} e^{-r(T-t) - \frac{d_+^2(b^{**}, S_t, t)}{2}}. \end{aligned}$$

where $N(\cdot)$ is the cumulative distribution function for standard normal distribution,

$$d_-(a,s,t) = \frac{1}{\theta\sqrt{T-t}}[-\ln a + \frac{\theta}{\sigma}(\frac{\mu+r-\sigma^2}{2}t - \ln\frac{s}{S_0}) + \frac{\theta^2}{2}(T-t)], \quad d_+(a,s,t) = -d_-(a,s,t),$$

and $\theta = \frac{\mu - r}{\sigma}$.

One-Constraint Optimization			Two-Constraint Optimization			
x_u	30	50	x_u	30	30	50
			z	20	25	25
			$\bar{z}$	28.8866	28.8866	45.5955
			z^*	18.8742	18.8742	18.8742
x^*	19.0670	19.0670	x^{**}	19.1258	19.5734	19.1434
a^*	14.5304	14.5304	a^{**}	14.3765	12.5785	14.1677
			b^{**}	0.0068	0.1326	0.0172
$CVaR(X_T^*)$	-15.2118	-15.2118	$CVaR(X_T^{**})$	-15.2067	-14.8405	-15.1483

Table 1: Black-Scholes' Example without & with Expected Return Constraint

We observe from Table 1 that different values of x_u do not have any impact in the one-constraint case as long as the optimal solution is achieved by the 'star-system' (x^*, a^*). Thus z^* calculated from this system is not impacted either. However, as the upper bound x_u increases, $\bar{z}$ increases, which allows more choices of higher expected return z.

Compare the results of the three cases where $x_u = 30$ in the table, we see that the higher the required return, the harder it is to obtain a low CVaR. This is also true for the two cases where $x_u = 50$. Now let us compare the two columns to the right: the two cases have the same required return 25. When the upper bound x_u is higher (=50), the attainable return $\bar{z}$ is higher (=45.5955), the required return $z = 25$ is relatively easier to achieve, thus has less impact in minimizing CVaR. In the two cases where $x_u = 50$, minimal CVaR only increases a little from -15.2118 to -15.1483 with the added return constraint. An intuition we can get from this comparison is that when we let the upper bound be extremely large ($x_u \uparrow \infty$), the attainable return $\bar{z}$ will also be so large that any required return z will seem to be effortless to obtain, thus the value of minimal CVaR is almost not impacted.

Let us have a look at the threshold b^{**}: when this number is small, the optimal of the two-constraint problem is very close to the optimal of the one-constraint problem. In the extreme that $b^{**} = 0$, the two problems coincide. With the same upper bound, the higher the required return, the more adjustment needs to be made on X_T^* to obtain X_T^{**}, thus the larger the b^{**} value. With the same required return, the higher the upper bound, the less the effort needed in adjusting X_T^*, thus the less the b^{**} value. In the limiting case, $x_u \uparrow \infty$, $b^{**} \downarrow 0$, thus $CVaR(X_T^{**}) \downarrow CVaR(X_T^*)$.

Figure 1 shows the efficient frontier of a mean-CVaR portfolio selection problem with bankruptcy prohibition and upper bound $x_u = 30$, where all the portfolios on the curve are efficient in the sense that the lowest risk as measured by CVaR is attained at each level of required expected terminal value z. The two dashed lines are at $z = z^* = 18.8742$ and $z = \bar{z} = 28.8866$. Given the parameters, the minimal CVaR is a constant for $z \le z^*$, and increases as the required z increases for $z \in (z^*, \bar{z}]$ between the dashed lines.

The star positioned at $(-x_r, x_r) = (-11.0517, 11.0517)$, where $x_r = X_0 e^{rT}$, is the portfolio that invests purely in money market account. Comparing to the traditional CML (i.e., capital market line) that shows the efficient frontier for a mean-variance portfolio selection problem, the pure money market account portfolio is no longer efficient.

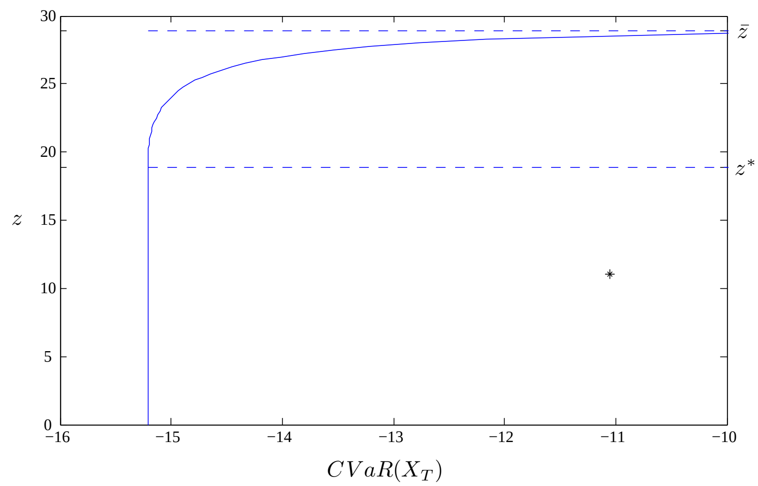

Figure 1: Efficient Frontier for Mean-CVaR Portfolio Selection

3 Proofs

Proof of Lemma 2.3. The problem of

$$\bar{z} = \max_{X \in \mathcal{F}} E[X] \quad s.t. \quad \tilde{E}[X] = x_r, \quad x_d \le X \le x_u \, a.s.$$

is equivalent to the Expected Shortfall Problem

$$\bar{z} == - \min_{X \in \mathcal{F}} E[(x_u - X)^+] \quad s.t. \quad \tilde{E}[X] = x_r, \quad X \ge x_d \, a.s..$$

Therefore, the answer is immediate. ◇

Proof of Lemma 2.4. Choose $x_d \leq x_1 < x_2 \leq x_r$. Let $X_1 = x_1 \mathbb{I}_{B_1} + x_u \mathbb{I}_{D_1}$ where $B_1 = \left\{\omega \in \Omega : \frac{d\tilde{P}}{dP}(\omega) \geq b_1\right\}$ and $D_1 = \left\{\omega \in \Omega : \frac{d\tilde{P}}{dP}(\omega) < b_1\right\}$. Choose b_1 such that $\tilde{E}[X_1] = x_r$. This capital constraint means $x_1\tilde{P}(B_1) + x_u\tilde{P}(D_1) = x_r$. Since $\tilde{P}(B_1) + \tilde{P}(D_1) = 1$, $\tilde{P}(B_1) = \frac{x_u - x_r}{x_u - x_1}$ and $\tilde{P}(D_1) = \frac{x_r - x_1}{x_u - x_1}$. Define $z_1 = E[X_1]$. Similarly, z_2, X_2, B_2, D_2, b_2 corresponds to x_2 where $b_1 > b_2$ and $\tilde{P}(B_2) = \frac{x_u - x_r}{x_u - x_2}$ and $\tilde{P}(D_2) = \frac{x_r - x_2}{x_u - x_2}$. Note that $D_2 \subset D_1$, $B_1 \subset B_2$ and $D_1 \backslash D_2 = B_2 \backslash B_1$. We have

$$
\begin{aligned}
z_1 - z_2 &= x_1 P(B_1) + x_u P(D_1) - x_2 P(B_2) - x_u P(D_2) \\
&= (x_u - x_2) P(B_2 \backslash B_1) - (x_2 - x_1) P(B_1) \\
&= (x_u - x_2) P\left(b_2 < \tfrac{d\tilde{P}}{dP}(\omega) < b_1\right) - (x_2 - x_1) P\left(\tfrac{d\tilde{P}}{dP}(\omega) \geq b_1\right) \\
&= (x_u - x_2) \int_{\left\{b_2 < \frac{d\tilde{P}}{dP}(\omega) < b_1\right\}} \frac{dP}{d\tilde{P}}(\omega) d\tilde{P}(\omega) - (x_2 - x_1) \int_{\left\{\frac{d\tilde{P}}{dP}(\omega) \geq b_1\right\}} \frac{dP}{d\tilde{P}}(\omega) d\tilde{P}(\omega) \\
&> (x_u - x_2) \frac{1}{b_1} \tilde{P}(B_2 \backslash B_1) - (x_2 - x_1) \frac{1}{b_1} \tilde{P}(B_1) \\
&= (x_u - x_2) \frac{1}{b_1} \left(\frac{x_u - x_r}{x_u - x_2} - \frac{x_u - x_r}{x_u - x_1}\right) - (x_2 - x_1) \frac{1}{b_1} \frac{x_u - x_r}{x_u - x_1} = 0.
\end{aligned}
$$

For any given $\epsilon > 0$, choose $x_2 - x_1 \leq \epsilon$, then

$$
\begin{aligned}
z_1 - z_2 &= (x_u - x_1) P(B_2 \backslash B_1) - (x_2 - x_1) P(B_2) \\
&\leq (x_u - x_1) P(B_2 \backslash B_1) \\
&\leq (x_u - x_1) \left(\frac{x_u - x_r}{x_u - x_2} - \frac{x_u - x_r}{x_u - x_1}\right) \\
&\leq \frac{(x_2 - x_1)(x_u - x_r)}{x_u - x_2} \leq x_2 - x_1 \leq \epsilon.
\end{aligned}
$$

Therefore, z decreases continuously as x increases when $x \in [x_d, x_r]$. When $x = x_d$, $z = \bar{z}$ from Definition 2.2. When $x = x_r$, $X \equiv x_r$ and $z = E[X] = x_r$. Similarly, we can show that z increases continuously from x_r to $\bar{z}$ as x increases from x_r to x_u. ◇

Lemma 2.6 and Proposition 2.7 are natural logical consequences and their proofs will be skipped.

Proof of Lemma 2.8. Choose $-\infty < b_1 < b_2 \leq \bar{b} = \bar{a} \leq a_2 < a_1 < \infty$. Let configuration $X_1 = x_d \mathbb{I}_{A_1} + x \mathbb{I}_{B_1} + x_u \mathbb{I}_{D_1}$ correspond to the pair (a_1, b_1) where $A_1 = \left\{\omega \in \Omega : \frac{d\tilde{P}}{dP}(\omega) > a_1\right\}, B_1 =$

$\left\{\omega \in \Omega : b_1 \leq \frac{d\tilde{P}}{dP}(\omega) \leq a_1\right\}, D_1 = \left\{\omega \in \Omega : \frac{d\tilde{P}}{dP}(\omega) < b_1\right\}$. Similarly, let configuration $X_2 = x_d \mathbb{I}_{A_2} + x \mathbb{I}_{B_2} + x_u \mathbb{I}_{D_2}$ correspond to the pair (a_2, b_2). Define $z_1 = E[X_1]$ and $z_2 = E[X_2]$. Since both X_1 and X_2 satisfy the capital constraint, we have

$$x_d \tilde{P}(A_1) + x\tilde{P}(B_1) + x_u \tilde{P}(D_1) = x_r = x_d \tilde{P}(A_2) + x\tilde{P}(B_2) + x_u \tilde{P}(D_2).$$

This simplifies to the equation

$$(x - x_d)\tilde{P}(A_2 \backslash A_1) = (x_u - x)\tilde{P}(D_2 \backslash D_1). \tag{11}$$

Then

$$\begin{aligned}
z_2 - z_1 &= x_d P(A_2) + xP(B_2) + x_u P(D_2) - x_d P(A_1) - xP(B_1) - x_u P(D_1) \\
&= (x_u - x)P(D_2 \backslash D_1) - (x - x_d)P(A_2 \backslash A_1) \\
&= (x_u - x)P(D_2 \backslash D_1) - (x_u - x)\frac{\tilde{P}(D_2 \backslash D_1)}{\tilde{P}(A_2 \backslash A_1)} P(A_2 \backslash A_1) \\
&= (x_u - x)\tilde{P}(D_2 \backslash D_1)\left(\frac{P(D_2 \backslash D_1)}{\tilde{P}(D_2 \backslash D_1)} - \frac{P(A_2 \backslash A_1)}{\tilde{P}(A_2 \backslash A_1)}\right) \\
&= (x_u - x)\tilde{P}(D_2 \backslash D_1)\left(\frac{\int_{\left\{b_1 \leq \frac{d\tilde{P}}{dP}(\omega) < b_2\right\}} \frac{dP}{d\tilde{P}}(\omega) d\tilde{P}(\omega)}{\tilde{P}(D_2 \backslash D_1)} - \frac{\int_{\left\{a_2 < \frac{d\tilde{P}}{dP}(\omega) \leq a_1\right\}} \frac{dP}{d\tilde{P}}(\omega) d\tilde{P}(\omega)}{\tilde{P}(A_2 \backslash A_1)}\right) \\
&\geq (x_u - x)\tilde{P}(D_2 \backslash D_1)\left(\frac{1}{b_2} - \frac{1}{a_2}\right) > 0.
\end{aligned}$$

Suppose the pair (a_1, b_1) is chosen so that X_1 satisfies the budget constraint $\tilde{E}[X_1] = x_r$. For any given $\epsilon > 0$, choose $b_2 - b_1$ small enough such that $P(D_2 \backslash D_1) \leq \frac{\epsilon}{x_u - x}$. Now choose a_2 such that $a_2 < a_1$ and equation (11) is satisfied. Then X_2 also satisfies the budget constraint $\tilde{E}[X_2] = x_r$, and

$$z_2 - z_1 = (x_u - x)P(D_2 \backslash D_1) - (x - x_d)P(A_2 \backslash A_1) \leq (x_u - x)P(D_2 \backslash D_1) \leq \epsilon.$$

We conclude that the expected value of the Three-Line configuration decreases continuously as b decreases and a increases. $\diamond$

PROOF OF PROPOSITION 2.9. Denote $\rho = \frac{d\tilde{P}}{dP}$. According to Lemma 2.8, there exists a Three-Line configuration $\hat{X} = x_d \mathbb{I}_A + x \mathbb{I}_B + x_u \mathbb{I}_D$ that satisfies the General Constraints:

$$E[X] = x_d P(A) + x P(B) + x_u P(D) = z,$$
$$\tilde{E}[X] = x_d \tilde{P}(A) + x \tilde{P}(B) + x_u \tilde{P}(D) = x_r.$$

where

$$A = \{\omega \in \Omega \,:\, \rho(\omega) > \hat{a}\}, \quad B = \left\{\omega \in \Omega \,:\, \hat{b} \le \rho(\omega) \le \hat{a}\right\}, \quad D = \left\{\omega \in \Omega \,:\, \rho(\omega) < \hat{b}\right\}.$$

As standard for convex optimization problems, if we can find a pair of Lagrange multipliers $\lambda \ge 0$ and $\mu \in \mathbb{R}$ such that $\hat{X}$ is the solution to the minimization problem

$$\inf_{X \in \mathcal{F},\, x_d \le X \le x_u} E[(x - X)^+ - \lambda X - \mu \rho X] = E[(x - \hat{X})^+ - \lambda \hat{X} - \mu \rho \hat{X}], \tag{12}$$

then $\hat{X}$ is the solution to the constrained problem

$$\inf_{X \in \mathcal{F},\, x_d \le X \le x_u} E[(x - X)^+], \quad s.t. \quad E[X] \ge z, \quad \tilde{E}[X] = x_r.$$

Define

$$\lambda = \frac{\hat{b}}{\hat{a} - \hat{b}}, \quad \mu = -\frac{1}{\hat{a} - \hat{b}}.$$

Then (12) becomes

$$\inf_{X \in \mathcal{F},\, x_d \le X \le x_u} E\left[(x - X)^+ + \tfrac{\rho - \hat{b}}{\hat{a} - \hat{b}} X\right].$$

Choose any $X \in \mathcal{F}$ where $x_d \le X \le x_u$, and denote $G = \{\omega \in \Omega \,:\, X(\omega) \ge x\}$ and $L = \{\omega \in \Omega \,:\, X(\omega) < x\}$.

Note that $\frac{\rho-\hat{b}}{\hat{a}-\hat{b}} > 1$ on set A, $0 \leq \frac{\rho-\hat{b}}{\hat{a}-\hat{b}} \leq 1$ on set B, $\frac{\rho-\hat{b}}{\hat{a}-\hat{b}} < 0$ on set D. Then the difference

$$
\begin{aligned}
& E\left[(x-X)^+ + \tfrac{\rho-\hat{b}}{\hat{a}-\hat{b}}X\right] - E\left[(x-\hat{X})^+ + \tfrac{\rho-\hat{b}}{\hat{a}-\hat{b}}\hat{X}\right] \\
&= E\left[(x-X)\mathbb{I}_L + \tfrac{\rho-\hat{b}}{\hat{a}-\hat{b}}X\left(\mathbb{I}_A+\mathbb{I}_B+\mathbb{I}_D\right)\right] - E\left[(x-x_d)\,\mathbb{I}_A + \tfrac{\rho-\hat{b}}{\hat{a}-\hat{b}}(x_d\mathbb{I}_A + x\mathbb{I}_B + x_u\mathbb{I}_D)\right] \\
&= E\left[(x-X)\mathbb{I}_L + \left(\tfrac{\rho-\hat{b}}{\hat{a}-\hat{b}}(X-x_d) - (x-x_d)\right)\mathbb{I}_A + \tfrac{\rho-\hat{b}}{\hat{a}-\hat{b}}\left(X-x\right)\mathbb{I}_B + \tfrac{\rho-\hat{b}}{\hat{a}-\hat{b}}\left(X-x_u\right)\mathbb{I}_D\right] \\
&\geq E\left[(x-X)\mathbb{I}_L + \left(X-x\right)\mathbb{I}_A + \tfrac{\rho-\hat{b}}{\hat{a}-\hat{b}}\left(X-x\right)\mathbb{I}_B + \tfrac{\rho-\hat{b}}{\hat{a}-\hat{b}}\left(X-x_u\right)\mathbb{I}_D\right] \\
&= E\left[(x-X)\left(\mathbb{I}_{L\cap A}+\mathbb{I}_{L\cap B}+\mathbb{I}_{L\cap D}\right) + (X-x)\left(\mathbb{I}_{A\cap G}+\mathbb{I}_{A\cap L}\right) + \tfrac{\rho-\hat{b}}{\hat{a}-\hat{b}}\left(X-x\right)\mathbb{I}_B + \tfrac{\rho-\hat{b}}{\hat{a}-\hat{b}}\left(X-x_u\right)\mathbb{I}_D\right] \\
&= E\left[(x-X)\left(\mathbb{I}_{L\cap B}+\mathbb{I}_{L\cap D}\right) + (X-x)\,\mathbb{I}_{A\cap G} + \tfrac{\rho-\hat{b}}{\hat{a}-\hat{b}}\left(X-x\right)\mathbb{I}_B + \tfrac{\rho-\hat{b}}{\hat{a}-\hat{b}}\left(X-x_u\right)\mathbb{I}_D\right] \\
&= E\left[(x-X)\left(\mathbb{I}_{L\cap B}+\mathbb{I}_{L\cap D}\right) + (X-x)\,\mathbb{I}_{A\cap G} + \tfrac{\rho-\hat{b}}{\hat{a}-\hat{b}}\left(X-x\right)\left(\mathbb{I}_{B\cap G}+\mathbb{I}_{B\cap L}\right) + \tfrac{\rho-\hat{b}}{\hat{a}-\hat{b}}\left(X-x_u\right)\left(\mathbb{I}_{D\cap G}+\mathbb{I}_{D\cap L}\right)\right] \\
&= E\left[(x-X)\left(1-\tfrac{\rho-\hat{b}}{\hat{a}-\hat{b}}\right)\mathbb{I}_{B\cap L} + \left(x-X+\tfrac{\rho-\hat{b}}{\hat{a}-\hat{b}}\left(X-x_u\right)\right)\mathbb{I}_{D\cap L} + (X-x)\,\mathbb{I}_{A\cap G}\right. \\
&\qquad\left. + \tfrac{\rho-\hat{b}}{\hat{a}-\hat{b}}\left(X-x\right)\mathbb{I}_{B\cap G} + \tfrac{\rho-\hat{b}}{\hat{a}-\hat{b}}\left(X-x_u\right)\mathbb{I}_{D\cap G}\right] \geq 0.
\end{aligned}
$$

The last inequality holds because each term inside the expectation is greater than or equal to zero. ◇

Theorem 2.10 is a direct consequence of Lemma 2.6, Proposition 2.7, and Proposition 2.9.

Proof of Lemma 2.13. The convexity of $v(x)$ is a simple consequence of its definition (4). Real-valued convex functions on $\mathbb{R}$ are continuous on its interior of the domain, so $v(x)$ is continuous on $\mathbb{R}$. ◇

Proof of Proposition 2.14. Obviously, **Case 2** dominates **Case 1** in the sense that its minimum is lower. In **Case 3**, by the continuity of $v(x)$, we have

$$\frac{1}{\lambda}\inf_{(x_{z1},x_{z2})}\left((x-x_d)P(A_x)-\lambda x\right) \leq \frac{1}{\lambda}\left((x_{z1}-x_d)P(A_{x_{z1}})-\lambda x_{z1}\right) = -x_{z1}.$$

The last equality comes from the fact $P(A_{x_{z1}})=0$: As in Lemma 2.8, we know that when $x = x_{z1}$, the Three-Line configuration $X = x_d\mathbb{I}_A + x\mathbb{I}_B + x_u\mathbb{I}_D$ degenerates to the Two-Line configuration $X = x_{z1}\mathbb{I}_B + x_u\mathbb{I}_D$

where $a_{x_{z1}} = \infty$. Therefore, **Case 3** dominates **Case 2**. In **Case 5**,

$$\begin{aligned}
\frac{1}{\lambda} \inf_{[x_u,\infty)} (v(x) - \lambda x) &= \frac{1}{\lambda} \inf_{[x_u,\infty)} \big((x - x_d)P(\bar{A}) + (x - x_u)P(\bar{B}) - \lambda x\big) \\
&= \frac{1}{\lambda} \inf_{[x_u,\infty)} \big((1-\lambda)x - x_d P(\bar{A}) - x_u P(\bar{B})\big) \\
&= \frac{1}{\lambda} \big((1-\lambda)x_u - x_d P(\bar{A}) - x_u P(\bar{B})\big) \\
&= \frac{1}{\lambda} \big((x_u - x_d)P(\bar{A}) - \lambda x_u\big) \\
&\geq \frac{1}{\lambda} \inf_{[x_{z2},x_u]} ((x - x_d)P(A_x) - \lambda x) \,.
\end{aligned}$$

Therefore, **Case 4** dominates **Case 5**. When $x \in [x_{z2}, x_u]$ and $\operatorname{ess\,sup} \frac{d\tilde{P}}{dP} > \frac{1}{\lambda}$, Theorem 2.10 and Theorem 2.11 imply that the infimum in **Case 4** is achieved either by $\bar{X}$ or X^*. Since we restrict $z \in (z^*, \bar{z}]$ where $z^* = \bar{z}$ by Definition 2.12 in the first case, we need not consider this case in the current proposition. In the second case, Lemma 2.4 implies that $x^* < x_{z2}$ (because $z > z^*$). By the convexity of $v(x)$, and then the continuity of $v(x)$,

$$\begin{aligned}
\frac{1}{\lambda} \inf_{[x_{z2},x_u]} ((x - x_d)P(A_x) - \lambda x) &= \frac{1}{\lambda} ((x_{z2} - x_d)P(A_{x_{z2}}) - \lambda x_{z2}) \\
&\geq \frac{1}{\lambda} \inf_{(x_{z1},x_{z2})} ((x - x_d)P(A_x) - \lambda x) \,.
\end{aligned}$$

Therefore, **Case 3** dominates **Case 4**. We have shown that **Case 3** actually provides the globally infimum:

$$\frac{1}{\lambda} \inf_{x \in \mathbb{R}} (v(x) - \lambda x) = \frac{1}{\lambda} \inf_{(x_{z1},x_{z2})} (v(x) - \lambda x).$$

Now we focus on $x \in (x_{z1}, x_{z2})$, where $X(x) = x_d \mathbb{I}_{A_x} + x \mathbb{I}_{B_x} + x_u \mathbb{I}_{D_x}$ satisfies the general constraints:

$$\begin{aligned}
E[X(x)] &= x_d P(A_x) + x P(B_x) + x_u P(D_x) = z, \\
\tilde{E}[X(x)] &= x_d \tilde{P}(A_x) + x \tilde{P}(B_x) + x_u \tilde{P}(D_x) = x_r,
\end{aligned}$$

and the definition for sets A_x, B_x and D_x are

$$A_x = \left\{\omega \in \Omega \,:\, \tfrac{d\tilde{P}}{dP}(\omega) > a_x\right\}, \quad B_x = \left\{\omega \in \Omega \,:\, b_x \leq \tfrac{d\tilde{P}}{dP}(\omega) \leq a_x\right\}, \quad D_x = \left\{\omega \in \Omega \,:\, \tfrac{d\tilde{P}}{dP}(\omega) < b_x\right\}.$$

Note that $v(x) = (x - x_d)P(A_x)$ (see Theorem 2.10). Since $P(A_x) + P(B_x) + P(D_x) = 1$ and $\tilde{P}(A_x) +$

$\tilde{P}(B_x) + \tilde{P}(D_x) = 1$, we rewrite the capital and return constraints as

$$x - z = (x - x_d)P(A_x) + (x - x_u)P(D_x),$$
$$x - x_r = (x - x_d)\tilde{P}(A_x) + (x - x_u)\tilde{P}(D_x).$$

Differentiating both sides with respect to x, we get

$$P(B_x) = (x - x_d)\frac{dP(A_x)}{dx} + (x - x_u)\frac{dP(D_x)}{dx},$$
$$\tilde{P}(B_x) = (x - x_d)\frac{d\tilde{P}(A_x)}{dx} + (x - x_u)\frac{d\tilde{P}(D_x)}{dx}.$$

Since

$$\frac{d\tilde{P}(A_x)}{dx} = a_x\frac{dP(A_x)}{dx}, \quad \frac{d\tilde{P}(D_x)}{dx} = b_x\frac{dP(D_x)}{dx},$$

we get

$$\frac{dP(A_x)}{dx} = \frac{\tilde{P}(B_x) - bP(B_x)}{(x - x_d)(a - b)}.$$

Therefore,

$$\begin{aligned}(v(x) - \lambda x)' &= P(A_x) + (x - x_d)\frac{dP(A_x)}{dx} - \lambda \\ &= P(A_x) + \frac{\tilde{P}(B_x) - bP(B_x)}{a - b} - \lambda.\end{aligned}$$

When the above derivative is zero, we arrive to the first order Euler condition

$$P(A_x) + \frac{\tilde{P}(B_x) - bP(B_x)}{a - b} - \lambda = 0.$$

To be precise, the above differentiation should be replaced by left-hand and right-hand derivatives as detailed in the Proof for Corollary 2.8 in Li and Xu [8]. But the first order Euler condition will turn out to be the same because we have assumed that the Radon-Nikodým derivative $\frac{d\tilde{P}}{dP}$ has continuous distribution.

To finish this proof, we need to show that there exists an $x \in (x_{z1}, x_{z2})$ where the first order Euler condition is satisfied. From Lemma 2.8, we know that as $x \searrow x_{z1}$, $a_x \nearrow \infty$, and $P(A_x) \searrow 0$. Therefore,

$$\lim_{x \searrow x_{z1}} (v(x) - \lambda x)' = -\lambda < 0.$$

As $x \nearrow x_{z2}$, $b_x \searrow 0$, and $P(D_x) \searrow 0$. Therefore,

$$\lim_{x \nearrow x_{z2}} (v(x) - \lambda x)' = P(A_{x_{z2}}) - \frac{\tilde{P}(A^c_{x_{z2}})}{a_{x_{z2}}} - \lambda.$$

This derivative coincides with the derivative of the value function of the Two-Line configuration that is optimal on the interval $x \in [x_{z2}, x_u]$ provided in Theorem 2.10 (see Proof for Corollary 2.8 in Li and Xu [8]). Again when $x \in [x_{z2}, x_u]$ and $\operatorname{ess\,sup} \frac{d\tilde{P}}{dP} > \frac{1}{\lambda}$, Theorem 2.10 and Theorem 2.11 imply that the infimum of $v(x) - \lambda x$ is achieved either by $\bar{X}$ or X^*. Since we restrict $z \in (z^*, \bar{z}]$ where $z^* = \bar{z}$ by Definition 2.12 in the first case, we need not consider this case in the current proposition. In the second case, Lemma 2.4 implies that $x^* < x_{z2}$ (because $z > z^*$). This in turn implies

$$P(A_{x_{z2}}) - \frac{\tilde{P}(A^c_{x_{z2}})}{a_{x_{z2}}} - \lambda < 0.$$

We have just shown that there exist some $x^{**} \in (x_{z1}, x_{z2})$ such that $(v(x) - \lambda x)'|_{x=x^{**}} = 0$. By the convexity of $v(x) - \lambda x$, this is the point where it obtains the minimum value. Now

$$\begin{aligned} CVaR(X^{**}) &= \frac{1}{\lambda}\left(v(x^{**}) - \lambda x^{**}\right) \\ &= \frac{1}{\lambda}\left((x^{**} - x_d)P(A^{**}) - \lambda x^{**}\right). \end{aligned}$$

$\diamond$

Proof of Theorem 2.15. Case 3 and 4 are already proved in Theorem 2.11 and Proposition 2.14. In Case 1 where $\operatorname{ess\,sup} \frac{d\tilde{P}}{dP} \le \frac{1}{\lambda}$ and $z = x_r$, $X = x_r$ is both feasible and optimal by Theorem 2.11. In Case 2, fix arbitrary $\epsilon > 0$. We will look for a Two-Line solution $X_\epsilon = x_\epsilon \mathbb{I}_{A_\epsilon} + \alpha_\epsilon \mathbb{I}_{B_\epsilon}$ with the right parameters $a_\epsilon, x_\epsilon, \alpha_\epsilon$ which satisfies both the capital constraint and return constraint:

$$E[X_\epsilon] = x_\epsilon P(A_\epsilon) + \alpha_\epsilon P(B_\epsilon) = z, \tag{13}$$

$$\tilde{E}[X_\epsilon] = x_\epsilon \tilde{P}(A_\epsilon) + \alpha_\epsilon \tilde{P}(B_\epsilon) = x_r, \tag{14}$$

where

$$A_\epsilon = \left\{\omega \in \Omega \,:\, \tfrac{d\tilde{P}}{dP}(\omega) > a_\epsilon\right\}, \quad B_\epsilon = \left\{\omega \in \Omega \,:\, \tfrac{d\tilde{P}}{dP}(\omega) \le a_\epsilon\right\},$$

and produces a CVaR level close to the lower bound:

$$CVaR(X_\epsilon) \leq CVaR(x_r) + \epsilon = -x_r + \epsilon.$$

First, we choose $x_\epsilon = x_r - \epsilon$. To find the remaining two parameters a_ϵ and α_ϵ so that equations (13) and (14) are satisfies, we note

$$x_r P(A_\epsilon) + x_r P(B_\epsilon) = x_r,$$
$$x_r \tilde{P}(A_\epsilon) + x_r \tilde{P}(B_\epsilon) = x_r,$$

and conclude that it is equivalent to find a pair of a_ϵ and α_ϵ such that the following two equalities are satisfied:

$$-\epsilon P(A_\epsilon) + (\alpha_\epsilon - x_r) P(B_\epsilon) = \gamma,$$
$$-\epsilon \tilde{P}(A_\epsilon) + (\alpha_\epsilon - x_r) \tilde{P}(B_\epsilon) = 0,$$

where we denote $\gamma = z - x_r$. If we can find a solution a_ϵ to the equation

$$\frac{\tilde{P}(B_\epsilon)}{P(B_\epsilon)} = \frac{\epsilon}{\gamma + \epsilon}, \tag{15}$$

then

$$\alpha_\epsilon = x_r + \frac{\tilde{P}(A_\epsilon)}{\tilde{P}(B_\epsilon)} \epsilon,$$

and we have the solutions for equations (13) and (14). It is not difficult to prove that the fraction $\frac{\tilde{P}(B)}{P(B)}$ increases continuously from 0 to 1 as a increases from 0 to $\frac{1}{\lambda}$. Therefore, we can find a solution $a_\epsilon \in (0, \frac{1}{\lambda})$ where (15) is satisfied. By definition (3),

$$CVaR_\lambda(X_\epsilon) = \frac{1}{\lambda} \inf_{x \in \mathbb{R}} \left(E[(x - X_\epsilon)^+] - \lambda x \right) \leq \frac{1}{\lambda} \left(E[(x_\epsilon - X_\epsilon)^+] - \lambda x_\epsilon \right) = -x_\epsilon.$$

The difference

$$CVaR_\lambda(X_\epsilon) - CVaR(x_r) \leq -x_\epsilon + x_r = \epsilon.$$

Under Assumption 1.1, the solution in Case 2 is almost surely unique, the result is proved. $\diamond$

Proof of Theorem 2.17. Case 1 and 3 are obviously true in light of Theorem 2.16. The proof for Case 2 is similar to that in the Proof of Theorem 2.15, so we will not repeat it here. Since $E[X^*] = z^* < z$ in case 4, $CVaR(X^*)$ is only a lower bound in this case. We first show that it is the true infimum obtained in Case 4. Fix arbitrary $\epsilon > 0$. We will look for a Three-Line solution $X_\epsilon = x_d \mathbb{I}_{A_\epsilon} + x_\epsilon \mathbb{I}_{B_\epsilon} + \alpha_\epsilon \mathbb{I}_{D_\epsilon}$ with the right parameters $a_\epsilon, b_\epsilon, x_\epsilon, \alpha_\epsilon$ which satisfies the general constraints:

$$E[X_\epsilon] = x_d P(A_\epsilon) + x_\epsilon P(B_\epsilon) + \alpha_\epsilon P(D_\epsilon) = z, \tag{16}$$

$$\tilde{E}[X_\epsilon] = x_d \tilde{P}(A_\epsilon) + x_\epsilon \tilde{P}(B_\epsilon) + \alpha_\epsilon \tilde{P}(D_\epsilon) = x_r, \tag{17}$$

where

$$A_\epsilon = \left\{\omega \in \Omega : \tfrac{d\tilde{P}}{dP}(\omega) > a_\epsilon\right\}, \quad B_\epsilon = \left\{\omega \in \Omega : b_\epsilon \le \tfrac{d\tilde{P}}{dP}(\omega) \le a_\epsilon\right\}, \quad D_\epsilon = \left\{\omega \in \Omega : \tfrac{d\tilde{P}}{dP}(\omega) < b_\epsilon\right\},$$

and produces a CVaR level close to the lower bound:

$$CVaR(X_\epsilon) \le CVaR(X^*) + \epsilon.$$

First, we choose $a_\epsilon = a^*$, $A_\epsilon = A^*$, $x_\epsilon = x^* - \delta$, where we define $\delta = \frac{\lambda}{\lambda - P(A^*)}\epsilon$. To find the remaining two parameters b_ϵ and α_ϵ so that equations (16) and (17) are satisfies, we note

$$E[X^*] = x_d P(A^*) + x^* P(B^*) = z^*,$$
$$\tilde{E}[X^*] = x_d \tilde{P}(A^*) + x^* \tilde{P}(B^*) = x_r,$$

and conclude that it is equivalent to find a pair of b_ϵ and α_ϵ such that the following two equalities are satisfied:

$$-\delta(P(B^*) - P(D_\epsilon)) + (\alpha_\epsilon - x^*)P(D_\epsilon) = \gamma,$$
$$-\delta(\tilde{P}(B^*) - \tilde{P}(D_\epsilon)) + (\alpha_\epsilon - x^*)\tilde{P}(D_\epsilon) = 0,$$

where we denote $\gamma = z - z^*$. If we can find a solution b_ϵ to the equation

$$\frac{\tilde{P}(D_\epsilon)}{P(D_\epsilon)} = \frac{\tilde{P}(B^*)}{\frac{\gamma}{\delta} + P(B^*)}, \tag{18}$$

then

$$\alpha_\epsilon = x^* + \left(\frac{\tilde{P}(B^*)}{\tilde{P}(D_\epsilon)} - 1 \right) \delta,$$

and we have the solutions for equations (16) and (17). It is not difficult to prove that the fraction $\frac{\tilde{P}(D)}{P(D)}$ increases continuously from 0 to $\frac{\tilde{P}(B^*)}{P(B^*)}$ as b increases from 0 to a^*. Therefore, we can find a solution $b_\epsilon \in (0, a^*)$ where (18) is satisfied. By definition (3),

$$\begin{aligned} CVaR_\lambda(X_\epsilon) &= \frac{1}{\lambda} \inf_{x \in \mathbb{R}} \left(E[(x - X_\epsilon)^+] - \lambda x \right) \\ &\leq \frac{1}{\lambda} \left(E[(x_\epsilon - X_\epsilon)^+] - \lambda x_\epsilon \right) \\ &= \frac{1}{\lambda}(x_\epsilon - x_d) P(A_\epsilon) - x_\epsilon. \end{aligned}$$

The difference

$$\begin{aligned} CVaR_\lambda(X_\epsilon) - CVaR(X^*) &\leq \frac{1}{\lambda}(x_\epsilon - x_d)P(A_\epsilon) - x_\epsilon - \frac{1}{\lambda}(x^* - x_d)P(A^*) + x^* \\ &= \frac{1}{\lambda}(x^* - x_d)(P(A_\epsilon) - P(A^*)) + \left(1 - \frac{P(A_\epsilon)}{\lambda} \right) (x^* - x_\epsilon) = \epsilon. \end{aligned}$$

Under Assumption 1.1, the solution in Case 4 is almost surely unique, the result is proved. ◇

4 Future Work

In Assumption 1.1, we require the Radon-Nikodým derivative to have continuous distribution. When this assumption is weakened, the results should still hold, albeit in a more complicated form. The out come resembles in the form of results obtained in Li and Xu [8]. It will also be very interesting to extend this result for CVaR minimization to minimizing Law-Invariant Risk Measures in general.

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