Optimal Acquisition of a Partially Hedgeable House

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Motivation

- Real estate is the main asset for most households
- Mostly absent in financial models or not included as a part of an optimization problem
- The optimization problem of the investors is usually on a finite time horizon.
- We consider the following problems:
  - Optimal housing purchase decision by a terminal time T
  - Interaction between the ownership of real estate and optimal portfolio allocation (both before and after buying the house)

Literature (economic side)

- Grossman and Laroque (1990) equilibrium model with a durable good
- Cocco (2005) calibrates the problem of an investor who chooses consumption, level of housing and optimal portfolio allocation
- Miao and Wang (2007) consider the optimal purchase decision when the cost of the asset is fixed (as a strike price) but not its price
- Cauley, Pavlov and Schwartz (2007) consider the optimal portfolio allocation problem of an investor who is already a homeowner and find the welfare impact of the housing constraint
- Tebaldi and Schwartz (2007) consider the problem of optimal portfolio allocation in the presence of illiquid assets

Literature (technical side)

- Cvitanić and Karatzas (1992) on optimal investment allocation with incomplete markets
- Karatzas and Wang (2001), who characterize the solution of mixed optimal stopping and control problems (as the one we consider in this paper)
- Brendle and Carmona (2004) and Hugonnier and Morellec (2007) (among others) consider the problem of hedging with incomplete markets

Our Problem

- An agent who maximizes utility from the final wealth (or the discounted one)
- Starts with a given level of wealth x
- Available financial assets are a risky stock and a (locally) risk-free bond
- There is also a house whose price is only partially correlated with the stock
  - According to Piazzesi, Schneider and Tuzel (2007) the correlation between the stock market and house prices is only 0.05
- The investor buys the house by a terminal time T (and holds it until T).
- There are financial incentives for buying the house (utility from the ownership, tax benefits,...)
- However, it can only be partially hedged (market incompleteness).

Our Model

- $W = [W^1 \quad W^2]'$ is a two dimensional standard Brownian motion (BM) process
- We assume all the standard good technical conditions are satisfied
- Risk-free asset: $\frac{dS_0}{S_0}(t) = r(t)dt$, with $r(t)$ the interest rate
- Stock price dynamics: $\frac{dS}{S}(t) = \mu(t)dt + \sigma(t)d\hat{W}(t)$

  where $\hat{W} = \rho W^1 + \sqrt{1-\rho^2}W^2$ with $-1 < \rho < 1$
- Financial Wealth: $X(0) = x$ and

  $dX = \pi\frac{dS}{S} + (X - \pi)rdt + Idt = [\pi(\mu - r) + rX + I]dt + \pi\sigma d\hat{W}$

  π is the amount of the wealth invested in the risky asset
  I is the net (of the consumption) income rate of the investor

Our Model (cont)

- There is a house whose price H satisfies
$$dH = H[\mu^H dt + \sigma^H dW^1]$$
- At some optimal time τ with $0 \leq \tau \leq T$, the investor decides to buy the house
  - The investor only has to pay $\delta H(\tau)$, $0 < \delta(\tau) < 1$
  - The balance, $(1-\delta)H(\tau)$, is the monetary value of owning the house, plus tax savings
  - We denote by $Y = X + H$ the wealth of the investor after buying the house
- The objective of the investor is to maximize CARA utility from final wealth
$$u(y) = -e^{-\gamma y}$$

Discussion of the Problem

- There is an incentive to buy the house early because of the addition to wealth
- However, after the house is bought, markets are incomplete
  - There is a component of wealth that cannot be hedged
  - It implies a welfare cost for the agent
- There is a trade-off between the two effects
- We use convex duality techniques to obtain the optimal wealth problem for fixed τ (we follow Brendle and Carmona 2004)
  - Does the convex duality work in an incomplete market?
  - In this case: YES, because of the CARA utility

The Solution

- The objective is to maximize $E[-e^{-\gamma Y(T)}]$ over all admissible pairs (τ, π), with τ optimal time of purchase
- We solve it in two steps:
  - First we solve
$$V^{\tau,x} = \sup_{\pi \in \mathcal{U}(\tau,T)} E_{\tau}^{\tau,x}[-e^{-\gamma Y(T)}]$$
    with $X(\tau) = x$
  - Then we solve for the optimal portfolio before buying the house
$$V^{\tau} = \sup_{\pi \in \mathcal{U}(0,\tau)} E[V^{\tau, X^{\pi}(\tau)}]$$
- The previous value function is equal to
$$\sup_{\substack{\pi \in \mathcal{U}(0,T) \\ X(\tau) = X(\tau_{-}) - \delta H(\tau)}} E[-e^{-\gamma Y(T)}]$$
  for fixed τ
- The optimal stopping time problem is, then
$$V = \sup_{0 \le \tau < T} V^{\tau}$$

Optimal Portfolio After Buying the House

- Assume, wlog, $r = 0$
- Define the following auxiliary process

$$L(s,t) = e^{-a[H(t)+\int_s^t l(u)du]-\int_s^t b(u)dW^1(u)-\int_s^t c(u)du}$$

with $a = \gamma(1-\rho^2)$, $b(t) = \frac{\mu\rho}{\sigma}(t)$ and $c(t) = \frac{\mu^2}{2\sigma^2}(t)$

- There exists a process ϕ such that

$$H(T) = \frac{1}{a}\{\int_\tau^T(\phi - b)(t)dW^1(t) + \int_\tau^T(\frac{1}{2}|\phi(u)|^2 - c)(t)dt - \ln E_\tau[L(\tau,T)]\} - \int_\tau^T l(u)du$$

- The value function is

$$V^{\tau,x} = -e^{-\gamma x}E_\tau[L(\tau,T)]^{\frac{1}{1-\rho^2}}$$

Optimal Portfolio After Buying the House (cont)

- And the optimal portfolio is

$$\pi^*(t) = \frac{\mu - \rho\sigma\phi}{\gamma(1-\rho^2)\sigma^2}(t)$$

- If all the model parameters are deterministic

$$\phi(t) = a\sigma^H(t)E_t[L(t,T)H(T)] + aE_t[L(t,T)\int_t^T \mathcal{D}_t I(u)du] + b(t)E_t[L(t,T)]$$

where $\mathcal{D}_t$ represents the Malliavin derivative

Optimal Portfolio Before Buying the House

- For fixed $\tau \in [0, T)$, we define the following two random variables

$$D(\tau) = -\delta H(\tau) - \frac{1}{a} \ln E_\tau[L(\tau, T)]$$

$$M(\tau) = e^{-a[D(\tau)+\int_0^\tau l(u)du]-\int_0^\tau b(u)dW^1(u)-\int_0^\tau c(u)du}$$

with a, b and c are as before

- There exists a process ψ such that

$$D(\tau) + \int_0^\tau l(u)du =$$

$$\frac{1}{a}\{\int_0^\tau (\psi - b)(t)dW^1 + \int_0^\tau (\frac{1}{2}\psi^2 - c)(t)dt - \ln E[M(\tau)]$$

- For fixed $\tau \in [0, T)$, the value function is

$$V^\tau = -e^{-\gamma x_0}(E[M(\tau)])^{\frac{1}{1-\rho^2}}$$

Optimal Portfolio Before Buying the House (cont)

- For fixed $\tau \in [0, T)$, the optimal portfolio before buying the house $\pi_*(t)$ is

$$\pi_*(t) = \frac{\mu - \rho\sigma\psi}{\gamma(1-\rho^2)\sigma^2}(t)$$

- When all the model parameters are deterministic

$$\psi(t) = -E_t[\mathcal{D}_t M(\tau]/M(t)$$

Optimal Stopping Time

- It is given by

$$V = \sup_{0 \le \tau < T} V^{\tau} = -e^{-\gamma x_0} \inf_{0 \le \tau < T} (E[M(\tau)])^{\frac{1}{1-\rho^2}}$$

- We can compute the expectation in the right hand side numerically by Monte Carlo simulation

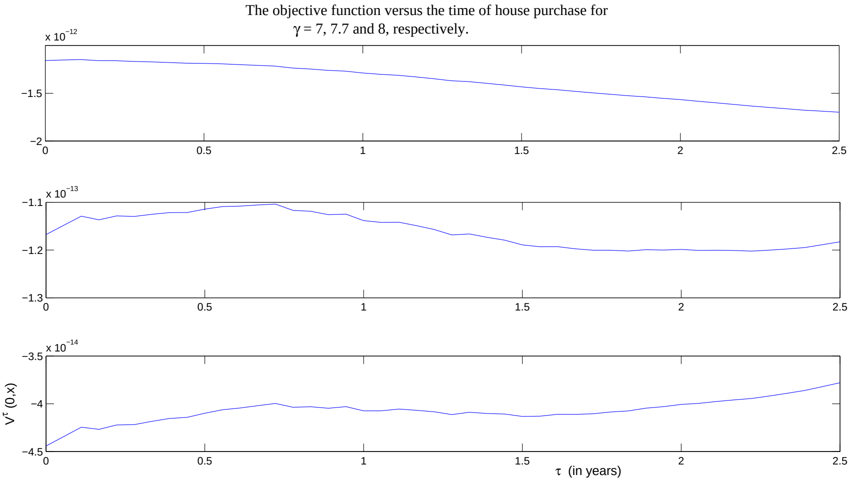
Numerical Exercise

- We look for parameter values for which it is optimal to buy the house immediately
- We focus on the effect of risk aversion, with everything else constant
- The state variable is h/x, or ratio of the house value to wealth
- The algorithm is as follows:
  - Set some parameter values, including a value for the coefficient of risk aversion γ and the state variable h/x
  - Find the value function for a grid of values for τ
  - If $\tau > 0$ change h/x
  - Stop when $\tau = 0$
  - Repeat the exercise for a different γ

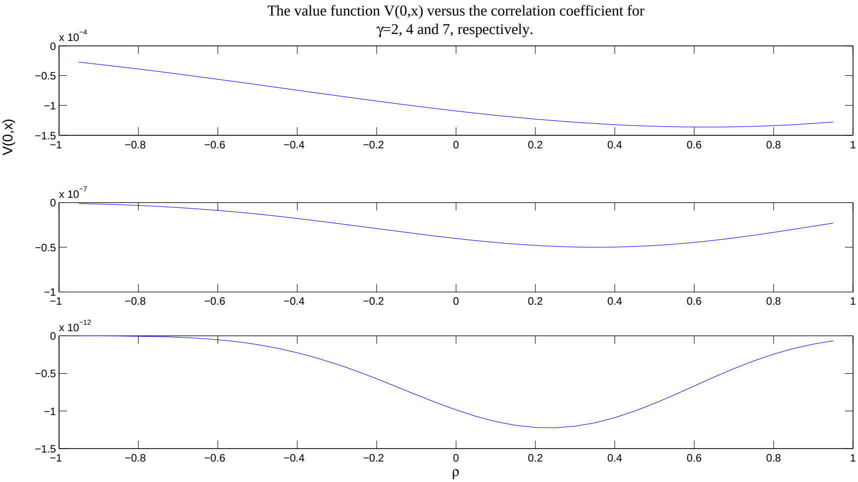
Numerical Exercise (cont)

- Parameter values
  - Asset parameters:
    $\mu = .11, \sigma = .26, \mu^H = .05, \sigma^H = .11, \rho = .1$
  - Horizon: $T = 2.5$
  - Cost of the house given by δ
    $$\delta(t) = .8 + .08t$$
  - Net income rate I
    $$I(t) = .35 + .04W^1(t)$$

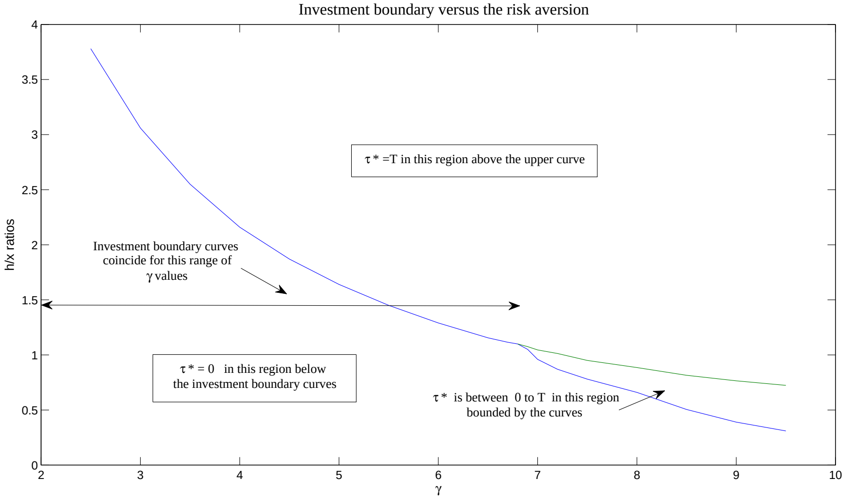
Value Function



Investment Boundary



Value Function with Drop in Income



Other Simple Extensions and Applications

- Random (Markovian) interest rate that is adapted to the filtration of W^1
- Different income rate process after buying the house (changes due to the retirement, rent, etc.)
- Trading a house (e.g. a smaller one) with another house (e.g. a larger one)
- Getting a lump sum income at time τ

provided that all the random processes above depend only on the same BM W^1