Hybrid Atlas Model
of financial equity market

Tomoyuki Ichiba ¹  Ioannis Karatzas ²,³  Adrian Banner ³
Vassilios Papathanakos ³  Robert Fernholz ³

¹ University of California, Santa Barbara  ² Columbia University, New York
³ INTECH, Princeton

November 2009
Outline

Introduction

Hybrid Atlas model
  Martingale Problem
  Stability
  Effective dimension
  Rankings
  Long-term growth relations

Portfolio analysis
  Stochastic Portfolio Theory
  Target portfolio
  Universal portfolio

Conclusion
Flow of Capital

Figure: Capital Distribution Curves (Percentage) for the S&P 500 Index of 1997 (Solid Line) and 1999 (Broken Line).
Figure: Capital distribution curves for 1929 (shortest curve) - 1999 (longest curve), every ten years. Source Fernholz('02).

What kind of models can describe this long-term stability?
A Model of Rankings [Hybrid Atlas model]

- Capital process \( X := \{(X_1(t), \ldots, X_n(t)) \, ; \, 0 \leq t < \infty \} \).
- Order Statistics:

\[
X(1)(t) \geq \cdots \geq X(n)(t) ; \quad 0 \leq t < \infty .
\]

Log capital \( Y := \log X \):

\[
Y(1)(t) \geq \cdots \geq Y(n)(t) ; \quad 0 \leq t < \infty .
\]

Dynamics of log capital:

\[
d Y_{(k)}(t) = (\gamma + \gamma_i + g_k) \, dt + \sigma_k \, d W_i(t) \quad \text{if} \quad Y_{(k)}(t) = Y_i(t) ;
\]

for \( 1 \leq i, k \leq n, \ 0 \leq t < \infty \), where \( W(\cdot) \) is \( n \)-dim. B. M.

<table>
<thead>
<tr>
<th>Drift (&quot;mean&quot;)</th>
<th>company name ( i )</th>
<th>( k ) th ranked company</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \gamma_i )</td>
<td>( \gamma_i )</td>
<td>( g_k )</td>
</tr>
<tr>
<td>( \sigma_k )</td>
<td>( \sigma_k &gt; 0 )</td>
<td></td>
</tr>
</tbody>
</table>

* Banner, Fernholz & Karatzas ('05), Chatterjee & Pal ('07, '09), Pal & Pitman ('08).
A Model of Rankings [Hybrid Atlas model]

- Capital process \( X := \{(X_1(t), \ldots, X_n(t)) : 0 \leq t < \infty \} \).
- Order Statistics:
  \[
  X(1)(t) \geq \cdots \geq X(n)(t) ; \quad 0 \leq t < \infty .
  \]

Log capital \( Y := \log X : \)
  \[
  Y(1)(t) \geq \cdots \geq Y(n)(t) ; \quad 0 \leq t < \infty .
  \]

Dynamics of log capital:
\[
d Y(k)(t) = (\gamma + \gamma_i + g_k) dt + \sigma_k d W_i(t) \quad \text{if} \quad Y(k)(t) = Y_i(t) ;
\]
for \( 1 \leq i, k \leq n, 0 \leq t < \infty \), where \( W(\cdot) \) is \( n \)-dim. B. M.

<table>
<thead>
<tr>
<th>Drift (&quot;mean&quot;)</th>
<th>company name ( i )</th>
<th>( k )th ranked company *</th>
</tr>
</thead>
<tbody>
<tr>
<td>Diffusion (&quot;variance&quot;)</td>
<td>( \gamma_i )</td>
<td>( g_k )</td>
</tr>
</tbody>
</table>

\( \sigma_k > 0 \)

* Banner, Fernholz & Karatzas (’05), Chatterjee & Pal (’07, ’09), Pal & Pitman (’08).
A Model of Rankings [Hybrid Atlas model]

- Capital process $X := \{ (X_1(t), \ldots, X_n(t)) \mid 0 \leq t < \infty \}$. 

- Order Statistics:
  
  $X_{(1)}(t) \geq \cdots \geq X_{(n)}(t) ; \ 0 \leq t < \infty$.

- Log capital $Y := \log X$:
  
  $Y_{(1)}(t) \geq \cdots \geq Y_{(n)}(t) ; \ 0 \leq t < \infty$.

**Dynamics of log capital:**

$$d Y_{(k)}(t) = (\gamma + \gamma_i + g_k) \, dt + \sigma_k \, d W_i(t) \quad \text{if } Y_{(k)}(t) = Y_i(t) ;$$

for $1 \leq i, k \leq n, \ 0 \leq t < \infty$, where $W(\cdot)$ is $n$–dim. B. M.

<table>
<thead>
<tr>
<th>company name $i$</th>
<th>$k$th ranked company $\ast$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Drift (“mean”)</td>
<td>$\gamma_i$</td>
</tr>
<tr>
<td>Diffusion (“variance”)</td>
<td>$g_k$</td>
</tr>
<tr>
<td></td>
<td>$\sigma_k &gt; 0$</td>
</tr>
</tbody>
</table>

$\ast$ Banner, Fernholz & Karatzas ('05), Chatterjee & Pal ('07, '09), Pal & Pitman ('08).
Illustration \((n = 3)\) of interactions through rank

Paths in \(\mathbb{R}_+ \times \text{Time}\). A path in different wedges of \(\mathbb{R}^n\).

Symmetric group \(\Sigma_n\) of permutations of \(\{1, \ldots, n\}\).
For \(n = 3\),
\(\{(1, 2, 3), (1, 3, 2), (2, 1, 3), (2, 3, 1), (3, 1, 2), (3, 2, 1)\}\).
Illustration \((n = 3)\) of interactions through rank

Paths in \(\mathbb{R}_+ \times \text{Time}\). A path in different wedges of \(\mathbb{R}^n\).

Symmetric group \(\Sigma_n\) of permutations of \(\{1, \ldots, n\}\).

For \(n = 3\),
\[\{(1, 2, 3), (1, 3, 2), (2, 1, 3), (2, 3, 1), (3, 1, 2), (3, 2, 1)\}\.]
Vector Representation

\[ d Y(t) = G(Y(t))d t + S(Y(t))dW(t) ; \quad 0 \leq t < \infty \]

\[ \Sigma_n : \text{symmetric group of permutations of } \{1, 2, \ldots, n\}. \]

For \( p \in \Sigma_n \) define wedges (chambers)

\[ \mathcal{R}_p := \{ x \in \mathbb{R}^n : x_p(1) \geq x_p(2) \geq \cdots \geq x_p(n) \} , \quad \mathbb{R}^n = \bigcup_{p \in \Sigma_n} \mathcal{R}_p , \]

(the inner points of \( \mathcal{R}_p \) and \( \mathcal{R}_{p'} \) are disjoint for \( p \neq p' \in \Sigma_n \)),

\[ Q_k^{(i)} := \{ x \in \mathbb{R}^n : x_i \text{ is ranked } k\text{th among } (x_1, \ldots, x_n) \} \]
\[ = \bigcup \{ p : p(k) = i \} \mathcal{R}_p ; \quad 1 \leq i, k \leq n , \]

\[ \bigcup_{j=1}^n Q_k^{(j)} = \mathbb{R}^n = \bigcup_{\ell=1}^n Q_\ell^{(i)} \text{ and } \mathcal{R}_p = \cap_{k=1}^n Q_k^{(p(k))} . \]

\[ G(y) = \sum_{p \in \Sigma_n} (g_{p^{-1}(1)} + \gamma_1 + \gamma, \ldots, g_{p^{-1}(n)} + \gamma_n + \gamma)' \cdot 1_{\mathcal{R}_p}(y) , \]

\[ S(y) = \sum_{p \in \Sigma_n} \text{diag}(\sigma_{p^{-1}(1)}, \ldots, \sigma_{p^{-1}(n)}) \cdot 1_{\mathcal{R}_p}(y) ; \quad y \in \mathbb{R}^n . \]
Vector Representation

\[ d \ Y(t) = G(Y(t)) \, dt + S(Y(t)) \, dW(t); \quad 0 \leq t < \infty \]

\( \Sigma_n \): symmetric group of permutations of \{1, 2, \ldots, n\}.

For \( p \in \Sigma_n \) define wedges (chambers)

\[ R_p := \{ x \in \mathbb{R}^n : x_{p(1)} \geq x_{p(2)} \geq \cdots \geq x_{p(n)} \}, \quad \mathbb{R}^n = \bigcup_{p \in \Sigma_n} R_p, \]

(the inner points of \( R_p \) and \( R_{p'} \) are disjoint for \( p \neq p' \in \Sigma_n \)),

\[ Q^{(i)}_k := \{ x \in \mathbb{R}^n : x_i \text{ is ranked } k \text{th among } (x_1, \ldots, x_n) \} \]

\[ = \bigcup \{ p : p(k) = i \} \, R_p; \quad 1 \leq i, k \leq n, \]

\[ \bigcup_{j=1}^n Q^{(j)}_k = \mathbb{R}^n = \bigcup_{\ell=1}^n Q^{(\ell)}_k \quad \text{and} \quad R_p = \bigcap_{k=1}^n Q^{(p(k))}_k. \]

\[ G(y) = \sum_{p \in \Sigma_n} \left( g_{p^{-1}(1)} + \gamma_1 + \gamma, \ldots, g_{p^{-1}(n)} + \gamma_n + \gamma \right)' \cdot 1_{R_p}(y), \]

\[ S(y) = \sum_{p \in \Sigma_n} \text{diag}(\sigma_{p^{-1}(1)}, \ldots, \sigma_{p^{-1}(n)}) \cdot 1_{R_p}(y); \quad y \in \mathbb{R}^n. \]
Theorem [Krylov('71), Stroock & Varadhan('79), Bass & Pardoux('87)] Suppose that the coefficients $G(\cdot)$ and $a(\cdot) := SS'(\cdot)$ are bounded and measurable, and that $a(\cdot)$ is uniformly positive-definite and piecewise constant in each wedge. For each $y_0 \in \mathbb{R}^n$ there is a unique one probability measure $\mathbb{P}$ on $C([0, \infty), \mathbb{R}^n)$ such that $\mathbb{P}(Y_0 = y_0) = 1$ and

$$f(Y_t) - f(Y_0) - \int_0^t L f(Y_s) \, ds; \quad 0 \leq t < \infty$$

is a $\mathbb{P}$ local martingale for every $f \in C^2(\mathbb{R}^2)$ where

$$L f(x) = \frac{1}{2} \sum_{i,j=1}^n a_{ij}(x) D_{ij} f(x) + \sum_{i=1}^n G_i(x) D_i f(x); \quad x \in \mathbb{R}^n.$$ 

This implies that the hybrid Atlas model is well-defined.
Market capitalization $X$ follows Hybrid Atlas model: the log capitalization $Y_i = \log X_i$ of company $i$ has

$$d Y_i(t) = \left( \gamma + \sum_{k=1}^{n} g_k 1_{Q_k(i)}(Y(t)) + \gamma_i \right) d t$$

$$+ \sum_{k=1}^{n} \sigma_k 1_{Q_k(i)}(Y(t)) d W_i(t); \quad 0 \leq t < \infty .$$
Model assumptions

Market capitalization $X$ follows Hybrid Atlas model: the log capitalization $Y_i = \log X_i$ of company $i$ has

$$\text{drift } \gamma + g_k + \gamma_i \text{ and volatility } \sigma_k,$$

when company $i$ is $k^{\text{th}}$ ranked, i.e., $Y \in Q_k^{(i)}$ for $1 \leq k, i \leq n$. Assume $\sigma_k > 0$, $(g_k, 1 \leq k \leq n)$, $(\gamma_i, 1 \leq i \leq n)$ and $\gamma$ are real constants with stability conditions

$$\sum_{k=1}^{n} g_k + \sum_{i=1}^{n} \gamma_i = 0, \quad \sum_{\ell=1}^{k} (g_\ell + \gamma_{p(\ell)}) < 0, \quad k = 1, \ldots, n-1, \ p \in \Sigma_n.$$

- $\gamma_i = 0, 1 \leq i \leq n$, $g_1 = \cdots = g_{n-1} = -g < 0$, $g_n = (n-1)g > 0$.
- $\gamma_i = 1 - (2i)/(n+1), 1 \leq i \leq n$, $g_k = -1, k = 1, \ldots, n-1, g_n = n - 1$. 
Model assumptions

Market capitalization $X$ follows Hybrid Atlas model: the log capitalization $Y_i = \log X_i$ of company $i$ has

$$\text{drift } \gamma + g_k + \gamma_i \text{ and volatility } \sigma_k,$$

when company $i$ is $k^{\text{th}}$ ranked, i.e., $Y \in Q_k^{(i)}$ for $1 \leq k, i \leq n$. Assume $\sigma_k > 0$, $(g_k, 1 \leq k \leq n)$, $(\gamma_i, 1 \leq i \leq n)$ and $\gamma$ are real constants with stability conditions

$$\sum_{k=1}^{n} g_k + \sum_{i=1}^{n} \gamma_i = 0, \quad \sum_{k}^{k} (g_\ell + \gamma_p(\ell)) < 0, \quad k = 1, \ldots, n-1, \ p \in \Sigma_n.$$ 

$\gamma_i = 0, 1 \leq i \leq n, \quad g_1 = \cdots = g_{n-1} = -g < 0, \quad g_n = (n-1)g > 0.$

$\gamma_i = 1 - (2i)/(n+1), 1 \leq i \leq n, \quad g_k = -1, k = 1, \ldots, n-1, \ g_n = n-1.$
Model assumptions

Market capitalization \( X \) follows Hybrid Atlas model: the log capitalization \( Y_i = \log X_i \) of company \( i \) has

\[
\text{drift } \gamma + g_k + \gamma_i \text{ and volatility } \sigma_k,
\]

when company \( i \) is \( k\text{-th} \) ranked, i.e., \( Y \in Q_k^{(i)} \) for \( 1 \leq k, i \leq n \). Assume \( \sigma_k > 0 \), \((g_k, 1 \leq k \leq n)\), \((\gamma_i, 1 \leq i \leq n)\) and \( \gamma \) are real constants with stability conditions

\[
\sum_{k=1}^{n} g_k + \sum_{i=1}^{n} \gamma_i = 0, \quad \sum_{\ell=1}^{k} (g_\ell + \gamma_{p(\ell)}) < 0, \quad k = 1, \ldots, n-1, \quad p \in \Sigma_n.
\]

\( \triangleright \) \( \gamma_i = 0 \), \( 1 \leq i \leq n \), \( g_1 = \cdots = g_{n-1} = -g < 0 \),
\( g_n = (n-1)g > 0 \).

\( \triangleright \) \( \gamma_i = 1 - (2i)/(n+1) \), \( 1 \leq i \leq n \),
\( g_k = -1 \), \( k = 1, \ldots, n-1 \), \( g_n = n - 1 \).
Model Summary

The log-capitalization $Y = \log X$ follows

$$d Y_i(t) = \left( \gamma + \sum_{k=1}^{n} g_k 1_{Q_k(i)}(Y(t)) + \gamma_i \right) dt$$

$$+ \sum_{k=1}^{n} \sigma_k 1_{Q_k(i)}(Y(t))d W_i(t); \quad 0 \leq t < \infty$$

where $\sigma_k > 0$, $(g_k, 1 \leq k \leq n)$, $(\gamma_i, 1 \leq i \leq n)$ and $\gamma$ are real constants with stability conditions

$$\sum_{k=1}^{n} g_k + \sum_{i=1}^{n} \gamma_i = 0, \quad \sum_{\ell=1}^{k} (g_\ell + \gamma_{p(\ell)}) < 0, \quad k = 1, \ldots, n-1, \quad p \in \Sigma_n.$$
Stochastic stability

The average $\bar{Y}(\cdot) := \sum_{i=1}^{n} Y_i(\cdot) / n$ of log-capitalization:

$$d\bar{Y}(t) = \gamma \, dt + \frac{1}{n} \sum_{k=1}^{n} \sigma_k \sum_{i=1}^{n} 1_{Q_k}^{(i)}(Y(t)) \, dW_i(t)$$

is a Brownian motion with variance rate $\sum_{k=1}^{n} \sigma_k^2 / n^2$ drift $\gamma$ by the Dambis-Dubins-Schwartz Theorem.

**Proposition** Under the assumptions the deviations $\tilde{Y}(\cdot) := (Y_1(\cdot) - \bar{Y}(\cdot), \ldots, Y_n(\cdot) - \bar{Y}(\cdot))$ from the average are stable in distribution, i.e., there is a unique invariant probability measure $\mu(\cdot)$ such that for every bounded, measurable function $f$ we have the Strong Law of Large Numbers

$$\lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} f(\tilde{Y}(t)) \, dt = \int_{\Pi} f(y) \mu(dy), \quad \text{a.s.}$$

where $\Pi := \{y \in \mathbb{R}^n : y_1 + \cdots + y_n = 0\}$.
Stochastic stability

The average $\overline{Y}(\cdot) := \sum_{i=1}^{n} Y_i(\cdot) / n$ of log-capitalization:

$$d\overline{Y}(t) = \gamma \, dt + \frac{1}{n} \sum_{k=1}^{n} \sigma_k \sum_{i=1}^{n} 1_{Q_k}^{(i)}(Y(t)) d W_i(t) + dB_k(t)$$

is a Brownian motion with variance rate $\sum_{k=1}^{n} \sigma_k^2 / n^2$ drift $\gamma$ by the Dambis-Dubins-Schwartz Theorem.

**Proposition** Under the assumptions the deviations $\tilde{Y}(\cdot) := (Y_1(\cdot) - \overline{Y}(\cdot), \ldots, Y_n(\cdot) - \overline{Y}(\cdot))$ from the average are stable in distribution, i.e., there is a unique invariant probability measure $\mu(\cdot)$ such that for every bounded, measurable function $f$ we have the Strong Law of Large Numbers

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T f(\tilde{Y}(t)) \, dt = \int_\Pi f(y) \mu(dy), \quad a.s.$$ 

where $\Pi := \{y \in \mathbb{R}^n : y_1 + \cdots + y_n = 0\}$. 
Average occupation times

Especially taking \( f(\cdot) = 1_{\mathcal{R}_p}(\cdot) \) or \( 1_{Q_k^{(i)}}(\cdot) \), we define from \( \mu \) the average occupation time of \( X \) in \( \mathcal{R}_p \) or \( Q_k^{(i)} \):

\[
\theta_p := \mu(\mathcal{R}_p) = \lim_{T \to \infty} \frac{1}{T} \int_0^T 1_{\mathcal{R}_p}(X(t)) \, dt \\
\theta_{k,i} := \mu(Q_k^{(i)}) = \lim_{T \to \infty} \frac{1}{T} \int_0^T 1_{Q_k^{(i)}}(X(t)) \, dt , \quad 1 \leq k, i \leq n ,
\]

since \( 1_{\mathcal{R}_p}(\tilde{Y}(\cdot)) = 1_{\mathcal{R}_p}(X(\cdot)) \) and \( 1_{Q_k^{(i)}}(X(\cdot)) = 1_{Q_k^{(i)}}(\tilde{Y}(\cdot)) \). By definition

\[
\uparrow \quad 0 \leq \theta_{k,i} = \sum_{\{p \in \Sigma_n : p(k) = i\}} \theta_p \leq 1 \quad \text{for} \quad 1 \leq k, i \leq n ,
\]

\[
\uparrow \quad \sum_{\ell=1}^n \theta_{\ell,i} = \sum_{j=1}^n \theta_{k,j} = 1 \quad \text{for} \quad 1 \leq k, i \leq n .
\]

What is the invariant distribution \( \mu \)?
Attainability

- One-dimensional Brownian motion attains the origin infinitely often.
- Two-dimensional Brownian motion does not attain the origin.

Does the process \( X(\cdot) \) attain the origin?

\[
X(t) = X(0) + \int_0^t b(X(s)) \, ds + \int_0^t \sigma(X(s)) \, dW(s)
\]

where \( b \) and \( \sigma \) are bounded measurable functions.

- Friedman('74), Bass & Pardoux('87).
Effective Dimension

Let us define *effective dimension* $ED(\cdot)$ by

$$ED(x) = \frac{\text{trace}(A(x))\|x\|^2}{x'A(x)x}; \quad x \in \mathbb{R}^n \setminus \{0\},$$

where $A(\cdot) = \sigma(\cdot)\sigma(\cdot)'$.

**Proposition**

Suppose $X(0) \neq 0$.

If $\inf_{x \in \mathbb{R}^n \setminus \{0\}} ED(x) \geq 2$, then $X(\cdot)$ does not attain the origin.

If $\sup_{x \in \mathbb{R}^n \setminus \{0\}} ED(x) < 2$ and if there is no drift, i.e., $b(\cdot) \equiv 0$, then $X(\cdot)$ attains the origin.

- Exterior Dirichlet Problem by Meyers and Serrin('60).
- Removal of drift by Girsanov’s theorem.
- If there is drift, take $\frac{[\text{trace}(A(x)) + x'b(x)]\cdot\|x\|^2}{x'A(x)x}$.
Now consider *triple collision*:

\[ \{ X_i(t) = X_j(t) = X_k(t) \text{ for some } t > 0, \ 1 \leq i < j < k \leq n \} . \]

**What is the probability of triple collision?**

Fix \( i = 1, j = 2, k = 3 \). Let us define the sum of squared distances:

\[ s^2(x) := (x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_3 - x_1)^2 = x'^{\top} D D' x ; \quad x \in \mathbb{R}^n , \]

where \((n \times 3)\) matrix \( D \) is defined by \( D := (d_1, d_2, d_3) \) with

\[
\begin{align*}
  d_1 &:= (1, -1, 0, \ldots, 0)' , \\
  d_2 &:= (0, 1, -1, 0, \ldots, 0)' , \\
  d_3 &:= (-1, 0, 1, \ldots, 0)' .
\end{align*}
\]

\( \mathcal{Z} := \{ x \in \mathbb{R}^n : s(x) = 0 \} . \)
Define the local effective dimension:

\[ R(x) := \frac{\text{trace}(D' A(x) D) \cdot x' DD' x}{x' DD' A(x) DD' x}; \quad x \in \mathbb{R}^n \setminus \mathbb{Z}. \]

**Proposition**

Suppose \( s(X(0)) \neq 0 \). If \( \inf_{x \in \mathbb{R}^n \setminus \mathbb{Z}} R(x) \geq 2 \), then

\[ \mathbb{P}(X_1(t) = X_2(t) = X_3(t) \text{ for some } t \geq 0) = 0. \]

If \( \sup_{x \in \mathbb{R}^n \setminus \mathbb{Z}} R(x) < 2 \) and if there is no drift, i.e., \( b(\cdot) \equiv 0 \), then

\[ \mathbb{P}(X_1(t) = X_2(t) = X_3(t) \text{ for some } t \geq 0) = 1. \]

- \( R(\cdot) \equiv 2 \) for \( n \)-dim. BM, i.e., \( A(\cdot) \equiv I \).
- If there is drift, take \( \frac{[\text{trace}(D' A(x) D) + x' DD' b(x)] \cdot x' DD' x}{x' DD' A(x) DD' x} \).

Idea of Proof: a comparison with Bessel process with dimension two.
Define the local effective dimension:

\[ R(x) := \frac{\text{trace}(D'A(x)D) \ x' DD'x}{x' DD' A(x) DD'x}; \quad x \in \mathbb{R}^n \setminus \mathcal{Z}. \]

---

**Proposition**

*Suppose* \( s(X(0)) \neq 0 \). *If* \( \inf_{x \in \mathbb{R}^n \setminus \mathcal{Z}} R(x) \geq 2 \), *then*

\[ \mathbb{P}(X_1(t) = X_2(t) = X_3(t) \text{ for some } t \geq 0) = 0. \]

*If* \( \sup_{x \in \mathbb{R}^n \setminus \mathcal{Z}} R(x) < 2 \) *and if there is no drift, i.e.,* \( b(\cdot) \equiv 0 \), *then*

\[ \mathbb{P}(X_1(t) = X_2(t) = X_3(t) \text{ for some } t \geq 0) = 1. \]

- \( R(\cdot) \equiv 2 \) for \( n \)-dim. BM, i.e., \( A(\cdot) \equiv I \).
- If there is drift, take \( \frac{\text{trace}(D'A(x)D) + x' DD'b(x)] \cdot x' DD'x}{x' DD' A(x) DD'x} \).

**Idea of Proof:** a comparison with Bessel process with dimension two.
Define the local effective dimension:

\[ R(x) := \frac{\text{trace}(D' A(x) D) x' DD' x}{x' DD' A(x) DD' x} ; \quad x \in \mathbb{R}^n \setminus \mathcal{Z}. \]

**Proposition**

Suppose \( s(X(0)) \neq 0 \). If \( \inf_{x \in \mathbb{R}^n \setminus \mathcal{Z}} R(x) \geq 2 \), then

\[ \mathbb{P}(X_1(t) = X_2(t) = X_3(t) \text{ for some } t \geq 0) = 0. \]

If \( \sup_{x \in \mathbb{R}^n \setminus \mathcal{Z}} R(x) < 2 \) and if there is no drift, i.e., \( b(\cdot) \equiv 0 \), then

\[ \mathbb{P}(X_1(t) = X_2(t) = X_3(t) \text{ for some } t \geq 0) = 1. \]

- \( R(\cdot) \equiv 2 \) for \( n-\)dim. BM, i.e., \( A(\cdot) \equiv I \).
- If there is drift, take \( \frac{[\text{trace}(D' A(x) D) + x' DD' b(x)] \cdot x' DD' x}{x' DD' A(x) DD' x} \).

**Idea of Proof:** a comparison with Bessel process with dimension two.
Rankings

Recall $Y(1)(\cdot) \geq Y(2)(\cdot) \geq \cdots \geq Y(n)(\cdot)$. Let us denote by $\Lambda^{k,j}(t)$ the local time accumulated at the origin by the nonnegative semimartingale $Y(k)(\cdot) - Y(j)(\cdot)$ up to time $t$ for $1 \leq k < j \leq n$.

**Theorem** [Banner & Ghomrasni (07)] For a general class of semimartingale $Y(\cdot)$, the rankings satisfy

$$
\begin{align*}
\text{d} Y(k)(t) &= \sum_{i=1}^{n} 1_{Q(i)}(Y(t)) \text{d} Y_i(t) \\
&+ (N_k(t))^{-1} \left[ \sum_{\ell=k+1}^{n} \text{d} \Lambda^{k,\ell}(t) - \sum_{\ell=1}^{k-1} \text{d} \Lambda^{\ell,k}(t) \right]
\end{align*}
$$

where $N_k(t)$ is the cardinality $|\{i : Y_i(t) = Y(k)(t)\}|$. 
Rankings

Recall \( Y_{(1)}(\cdot) \geq Y_{(2)}(\cdot) \geq \cdots \geq Y_{(n)}(\cdot) \). Let us denote by \( \Lambda^{k,j}(t) \) the local time accumulated at the origin by the nonnegative semimartingale \( Y_{(k)}(\cdot) - Y_{(j)}(\cdot) \) up to time \( t \) for \( 1 \leq k < j \leq n \).

**Lemma** Under the non-degeneracy condition \( \sigma_k > 0 \) for \( k = 1, \ldots, n \),

\[
dY_{(k)}(t) = \left( \gamma + g_k + \sum_{i=1}^{n} \gamma_i 1_{Q_k(i)}(Y(t)) \right) dt + \sigma_k dB_k(t)
\]

\[
+ \frac{1}{2} \left[ d\Lambda^{k,k+1}(t) - d\Lambda^{k-1,k}(t) \right].
\]

for \( k = 1, \ldots, n \), \( 0 \leq t \leq T \).

Idea of Proof: a comparison with a Bessel process with dimension one to show \( \Lambda^{k,\ell}(\cdot) \equiv 0 , |k - \ell| \geq 2 \).
Recall $Y_{(1)}(\cdot) \geq Y_{(2)}(\cdot) \geq \cdots \geq Y_{(n)}(\cdot)$. Let us denote by $\wedge_{k,j}(t)$ the local time accumulated at the origin by the nonnegative semimartingale $Y_{(k)}(\cdot) - Y_{(j)}(\cdot)$ up to time $t$ for $1 \leq k < j \leq n$.

**Lemma** Under the non-degeneracy condition $\sigma_k > 0$ for $k = 1, \ldots, n$,

$$dY_{(k)}(t) = \left(\gamma + g_k + \sum_{i=1}^{n} \gamma_i^1 Q_k^{(i)}(Y(t))\right) dt + \sigma_k \, dB_k(t)$$

$$+ \frac{1}{2} \left[ d\wedge_{k,k+1}(t) - d\wedge_{k-1,k}(t) \right].$$

for $k = 1, \ldots, n$, $0 \leq t \leq T$.

**Idea of Proof:** a comparison with a Bessel process with dimension one to show $\wedge_{k,\ell}(\cdot) \equiv 0$, $|k - \ell| \geq 2$. 
Long-term growth relations

**Proposition** Under the assumptions we obtain the following long-term growth relations:

\[
\lim_{T \to \infty} \frac{Y_i(T)}{T} = \lim_{T \to \infty} \frac{\log X_i(T)}{T} = \gamma = \lim_{T \to \infty} \frac{\log \sum_{i=1}^{n} X_i(T)}{T} \quad \text{a.s.}
\]

Thus the model is coherent:

\[
\lim_{T \to \infty} \frac{1}{T} \log \mu_i(T) = 0 \quad \text{a.s.;} \quad i = 1, \ldots, n
\]

where \( \mu_i(\cdot) = \frac{X_i(\cdot)}{X_1(\cdot) + \cdots + X_n(\cdot)} \). Moreover,

\[
\sum_{k=1}^{n} g_k \theta_{k,i} + \gamma_i = 0 ; \quad i = 1, \ldots, n.
\]
\[ \sum_{k=1}^{n} g_{k} \theta_{k,i} + \gamma_{i} = 0; \quad i = 1, \ldots, n. \]

The log-capitalization \( Y \) grows with rate \( \gamma \) and follows

\[ d Y_{i}(t) = \left( \gamma + \sum_{k=1}^{n} g_{k} 1_{Q_{k}^{(i)}}(Y(t)) + \gamma_{i} \right) dt \]

\[ + \sum_{k=1}^{n} \sigma_{k} 1_{Q_{k}^{(i)}}(Y(t))d W_{i}(t); \quad 0 \leq t < \infty \]

for \( i = 1, \ldots, n \). The ranking \( (Y_{(1)}(\cdot), \ldots, Y_{(n)}(\cdot)) \) follows

\[ dY_{(k)}(t) = \left( \gamma + g_{k} + \sum_{i=1}^{n} \gamma_{i} 1_{Q_{k}^{(i)}}(Y(t)) \right) dt + \sigma_{k} d B_{k}(t) \]

\[ + \frac{1}{2} \left[ d \Lambda^{k,k+1}(t) - d \Lambda^{k-1,k}(t) \right]. \]

for \( k = 1, \ldots, n, \; 0 \leq t < \infty \).
Semimartingale reflected Brownian motions

The adjacent differences (gaps) $\Xi(\cdot) := (\Xi_1(\cdot), \ldots, \Xi_n(\cdot))'$ where $\Xi_k(\cdot) := Y_{(k)}(\cdot) - Y_{(k+1)}(\cdot)$ for $k = 1, \ldots, n-1$ can be seen as a semimartingale reflected Brownian motion (SRBM):

$$\Xi(t) = \Xi(0) + \zeta(t) + (I_n - Q)\Lambda(t)$$

where $\zeta(\cdot) := (\zeta_1(\cdot), \ldots, \zeta_n(\cdot))'$, $\Lambda(\cdot) := (\Lambda^{1,2}(\cdot), \ldots, \Lambda^{n-1,n}(\cdot))'$,

$$\zeta_k(\cdot) := \sum_{i=1}^{n} \int_{0}^{\cdot} \mathbf{1}_{Q_k(i)}(Y(s)) \, dY(s) - \sum_{i=1}^{n} \int_{0}^{\cdot} \mathbf{1}_{Q_{k+1}(i)}(Y(s)) \, dY(s)$$

for $k = 1, \ldots, n-1$, and $Q$ is an $(n-1) \times (n-1)$ matrix with elements
Thus the gaps $\Xi_k := Y_{(k)}(\cdot) - Y_{(k+1)}(\cdot)$ follow

$$\Xi(t) = \Xi(0) + \zeta(t) + (I_n - Q)\Lambda(t)$$

In order to study the invariant measure $\mu$, we apply the theory of semimartingale reflected Brownian motions developed by M. Harrison, M. Reiman, R. Williams and others.

In addition to the model assumptions, we assume linearly growing variances:

$$\sigma_2^2 - \sigma_1^2 = \sigma_3^2 - \sigma_2^2 = \cdots = \sigma_n^2 - \sigma_{n-1}^2.$$
Invariant distribution of gaps and index

Let us define the indicator map $\mathbb{R}^n \ni x \mapsto p^x \in \Sigma_n$ such that $x_{p^x(1)} \geq x_{p^x(2)} \geq \cdots \geq x_{p(n)}$, and the index process $\mathcal{P}_t := p^{Y(t)}$.

**Proposition** Under the stability and the linearly growing variance conditions the invariant distribution $\nu(\cdot)$ of $(\Xi(\cdot), \mathcal{P})$ is

$$\nu(A \times B) = \left( \sum_{q \in \Sigma_n} \prod_{k=1}^{n-1} \lambda_{q,k}^{-1} \right)^{-1} \sum_{p \in \Sigma_n} \int_A \exp(-\langle \lambda_p, z \rangle) d z$$

for every measurable set $A \times B$ where $\lambda_p := (\lambda_{p,1}, \ldots, \lambda_{p,n-1})'$ is the vector of components

$$\lambda_{p,k} := \frac{-4\left( \sum_{\ell=1}^{k} g_{\ell} + \gamma_{p(\ell)} \right)}{\sigma_k^2 + \sigma_{k+1}^2} > 0; \quad p \in \Sigma_n, \quad 1 \leq k \leq n-1.$$

Proof: an extension from M. Harrison and R. Williams ('87).
Corollary The average occupation times are

\[
\theta_p = \left( \sum_{q \in \Sigma_n} \prod_{k=1}^{n-1} \lambda_{q,k}^{-1} \right)^{-1} \prod_{j=1}^{n-1} \lambda_{p,j}^{-1}
\]

and

\[
\theta_{k,i} = \sum_{\{p \in \Sigma_n : p(k) = i\}} \theta_p
\]

for \( p \in \Sigma_n \) and \( 1 \leq k, i \leq n \).

If all \( \gamma_i = 0 \) and \( \sigma_1^2 = \cdots = \sigma_n^2 \), then

\[
\theta_{k,i} = \frac{1}{n}
\]

for \( 1 \leq k, i \leq n \).

Heat map of \( \theta_{k,i} \) when \( n = 10 \), \( \sigma_k^2 = 1 + k \), \( g_k = -1 \) for \( k = 1, \ldots, 9 \), \( g_{10} = 9 \), and

\[
\gamma_i = 1 - (2i)/(n+1)
\]

for \( i = 1, \ldots, n \).
Market weights come from Pareto type

**Corollary** The joint invariant distribution of market shares

\[ \mu(i)(\cdot) := \frac{X(i)(\cdot)}{X(1)(\cdot) + \cdots + X(n)(\cdot)} ; \quad i = 1, \ldots, n \]

has the density

\[
\varphi(m_1, \ldots, m_{n-1}) = \sum_{p \in \Pi_n} \theta_p \frac{\lambda_{p,1} \cdots \lambda_{p,n-1}}{m_1^{\lambda_{p,1}+1} \cdot m_2^{\lambda_{p,2}-\lambda_{p,1}+1} \cdots m_{n-1}^{\lambda_{p,n-1}-\lambda_{p,n-2}+1} m_n^{\lambda_{p,n-2}+1}} ,
\]

\[ 0 < m_n \leq m_{n-1} \leq \ldots \leq m_1 < 1 , \]

\[ m_n = 1 - m_1 - \cdots - m_{n-1} . \]

This is a distribution of ratios of Pareto type distribution.
Expected capital distribution curves

From the expected slopes
$$E^\nu \left[ \frac{\log \mu(k) - \log \mu(k-1)}{\log(k+1) - \log k} \right] = -\frac{E^\nu(\Xi_k)}{\log(1+k^{-1})}$$
we obtain expected capital distribution curves.

$n = 5000, \ g_n = c_*(2n - 1), \ g_k = 0, \ 1 \leq k \leq n - 1, \ \gamma_1 = -c_*, \ \gamma_i = -2c_*, \ 2 \leq i \leq n, \ \sigma_k^2 = 0.075 + 6k \times 10^{-5}, \ 1 \leq k \leq n. (i) \ c_* = 0.02, \ (ii) \ c_* = 0.03, \ (iii) \ c_* = 0.04.$

(iv) $c_* = 0.02, \ g_1 = -0.016, \ g_k = 0, \ 2 \leq k \leq n - 1, \ g_n = (0.02)(2n - 1) + 0.016,$

(v) $g_1 = \cdots = g_{50} = -0.016, \ g_k = 0, \ 51 \leq k \leq n - 1, \ g_n = (0.02)(2n - 1) + 0.8.$
Empirical data


Source: Fernholz ('02)

Capital Stocks and Portfolio Rules

Market $X = ((X_1(t), \ldots, X_n(t)), t \geq 0)$ of $n$ companies

$$
\log \frac{X_i(T)}{X_i(0)} = \int_0^T G_i(t) dt + \int_0^T \sum_{\nu=1}^n S_{i,\nu}(t) dW_\nu(t),
$$

with initial capital $X_i(0) = x_i > 0$, $i = 1, \ldots, n$, $0 \leq T < \infty$.

Define $a_{ij}(\cdot) = \sum_{\nu=1}^n S_{i,\nu}(\cdot)S_{j,\nu}(\cdot)$ and $A_{ij}(\cdot) = \int_0^\cdot a_{ij}(t) dt$.

Long only Portfolio rule $\pi$ and its wealth $V^{\pi}$.

Choose $\pi \in \Delta^n_+ := \{ x \in \mathbb{R}^n : \sum x_i = 1, x_i \geq 0 \}$

invest $\pi_i V^{\pi}$ of money to company $i$ for $i = 1, \ldots, n$, i.e., $\pi_i V^{\pi}/X_i$ share of company $i$:

$$
dV^{\pi}(t) = \sum_{i=1}^n \frac{\pi_i(t) V^{\pi}(t)}{X_i(t)} dX_i(t), \quad 0 \leq t < \infty,
$$

$V^{\pi}(0) = w$. 
Capital Stocks and Portfolio Rules

- Market \( X = ((X_1(t), \ldots, X_n(t)), t \geq 0) \) of \( n \) companies

\[
\log \frac{X_i(T)}{X_i(0)} = \int_0^T G_i(t)\,dt + \int_0^T \sum_{\nu=1}^n S_{i,\nu}(t)\,dW_{\nu}(t),
\]

with initial capital \( X_i(0) = x_i > 0, \ i = 1, \ldots, n, \ 0 \leq T < \infty \).

Define \( a_{ij}(\cdot) = \sum_{\nu=1}^n S_{i,\nu}(\cdot)S_{j,\nu}(\cdot) \) and \( A_{ij}(\cdot) = \int_0^\cdot a_{ij}(t)\,dt \).

- Long only Portfolio rule \( \pi \) and its wealth \( V^\pi \).

Choose \( \pi \in \Delta^n_+ := \{ x \in \mathbb{R}^n : \sum x_i = 1, \ x_i \geq 0 \} \)

invest \( \pi_i V^\pi \) of money to company \( i \) for \( i = 1, \ldots, n \), i.e., \( \pi_i V^\pi / X_i \) share of company \( i \):

\[
dV^\pi(t) = \sum_{i=1}^n \frac{\pi_i(t)V^\pi(t)}{X_i(t)}dX_i(t), \quad 0 \leq t < \infty,
\]

\( V^\pi(0) = w \).
Capital Stocks and Portfolio Rules

Market $X = ((X_1(t), \ldots, X_n(t)), t \geq 0)$ of $n$ companies

$$\log \frac{X_i(T)}{X_i(0)} = \int_0^T G_i(t)dt + \int_0^T \sum_{\nu=1}^n S_{i,\nu}(t)dW_{\nu}(t),$$

with initial capital $X_i(0) = x_i > 0$, $i = 1, \ldots, n$, $0 \leq T < \infty$. Define $a_{ij}(\cdot) = \sum_{\nu=1}^n S_{i\nu}(\cdot)S_{j\nu}(\cdot)$ and $A_{ij}(\cdot) = \int_0^\cdot a_{ij}(t)dt$.

Long only Portfolio rule $\pi$ and its wealth $V^\pi$.

Choose $\pi \in \Delta^n_+ := \{x \in \mathbb{R}^n : \sum x_i = 1, x_i \geq 0\}$ invest $\pi_i V^\pi$ of money to company $i$ for $i = 1, \ldots, n$, i.e., $\pi_i V^\pi / X_i$ share of company $i$:

$$dV^\pi(t) = \sum_{i=1}^n \frac{\pi_i(t) V^\pi(t)}{X_i(t)}dX_i(t), \quad 0 \leq t < \infty,$$

$V^\pi(0) = w$. 


Portfolios and Relative Arbitrage

- **Market portfolio**: Take
  \[ \pi(t) = m(t) = (m_1(t), \cdots, m_n(t)) \in \Delta^n \] where
  \[ m_i(t) = \frac{X_i(t)}{X_1 + \cdots + X_n(t)}, \quad i = 1, \ldots, n, \quad 0 \leq t < \infty. \]

- **Diversity weighted portfolio**: Given \( p \in [0, 1] \), take
  \[ \pi_i(t) = \frac{(m_i(t))^p}{\sum_{j=1}^n (m_j(t))^p} \quad \text{for} \quad i = 1, \ldots, n, \quad 0 \leq t < \infty. \]

- **Functionally generated portfolio** (Fernholz (’02) & Karatzas (’08)).

- **A portfolio** \( \pi \) **represents an arbitrage opportunity** relative to another portfolio \( \rho \) on \( [0, T] \), if
  \[ \mathbb{P}(V^{\pi}(T) \geq V^{\rho}(T)) = 1, \quad \mathbb{P}(V^{\pi}(T) > V^{\rho}(T)) > 0. \]

*Can we find an arbitrage opportunity \( \pi \) relative to \( m \)?
Portfolios and Relative Arbitrage

- **Market portfolio**: Take
  \[ \pi(t) = m(t) = (m_1(t), \ldots, m_n(t)) \in \Delta^n \] where
  \[ m_i(t) = \frac{X_i(t)}{X_1(t) + \cdots + X_n(t)}, \quad i = 1, \ldots, n, \quad 0 \leq t < \infty. \]

- **Diversity weighted portfolio**: Given \( p \in [0, 1] \), take
  \[ \pi_i(t) = \frac{(m_i(t))^p}{\sum_{j=1}^n (m_j(t))^p} \quad \text{for} \quad i = 1, \ldots, n, \quad 0 \leq t < \infty. \]

- **Functionally generated portfolio** (Fernholz ('02) & Karatzas ('08)).

- **A portfolio** \( \pi \) represents an arbitrage opportunity relative to another portfolio \( \rho \) on \([0, T]\), if
  \[ \mathbb{P}(V^{\pi}(T) \geq V^{\rho}(T)) = 1, \quad \mathbb{P}(V^{\pi}(T) > V^{\rho}(T)) > 0. \]

Can we find an arbitrage opportunity \( \pi \) relative to \( m \)?
Constant-portfolio

For a constant-proportion $\pi(\cdot) \equiv \pi$,

$$V^{\pi}(t) = w \cdot \exp \left[ \sum_{i=1}^{n} \pi_i \left\{ \frac{A_{ij}(t)}{2} + \log \left( \frac{X_i(t)}{X_i(0)} \right) \right\} - \frac{1}{2} \sum_{i,j=1}^{n} \pi_i A_{ij}(t) \pi_j \right]$$

for $0 \leq t < \infty$.

Here $A_{ij}(\cdot) = \int_{0}^{\cdot} a_{ij}(t) dt$ and $a_{ij}(\cdot) = \sum_{\nu=1}^{n} S_{i\nu}(\cdot) S_{j\nu}(\cdot)$,

$$d \ Y(t) = G(Y(t))dt + S(Y(t))dW(t) ; \ 0 \leq t < \infty,$$

$$G(y) = \sum_{p \in \Sigma_n} (g_{p}^{-1}(1) + \gamma_1 + \gamma, \ldots, g_{p}^{-1}(n) + \gamma n + \gamma)' \cdot 1_{\mathbb{R}^p}(y) ,$$

$$S(y) = \sum_{p \in \Sigma_n} \text{diag}(\sigma_{p}^{-1}(1), \ldots, \sigma_{p}^{-1}(n)) \cdot 1_{\mathbb{R}^p}(y) ; \ y \in \mathbb{R}^n.$$
Target Portfolio (Cover (’91) & Jamshidian (’92))

\[ V^\pi(\cdot) = w \cdot \exp \left[ \sum_{i=1}^{n} \pi_i \left\{ \frac{A_{ii}(t)}{2} + \log \left( \frac{X_i(\cdot)}{X_i(0)} \right) \right\} - \frac{1}{2} \sum_{i,j=1}^{n} \pi_i A_{ij}(\cdot) \pi_j \right] \]

Target Portfolio \( \Pi^*(t) \) maximizes the wealth \( V^\pi(t) \) for \( t \geq 0 \):

\[ V_*(t) := \max_{\pi \in \Delta_n^+} V^\pi(t), \quad \Pi^*(t) := \arg \max_{\pi \in \Delta_n^+} V^\pi(t), \]

where by Lagrange method we obtain

\[
\Pi^*_i(t) = \left( 2A_{ii}(t) \sum_{j=1}^{n} \frac{1}{A_{jj}(t)} \right)^{-1} \left[ 2 - n - 2 \sum_{j=1}^{n} \frac{1}{A_{jj}(t)} \log \left( \frac{X_j(t)}{X_j(0)} \right) \right] \\
+ \frac{1}{2} + \frac{1}{A_{ii}(t)} \log \left( \frac{X_i(t)}{X_i(0)} \right) ; \quad 0 \leq t < \infty .
\]
Asymptotic Target Portfolio

Under the hybrid Atlas model with the assumptions

\[ v(\pi) := \lim_{T \to \infty} \frac{1}{T} \log V^\pi(T) = \gamma + \frac{1}{2} \left( \sum_{i=1}^{n} \pi_i a_{ii}^\infty - \sum_{i=1}^{n} \pi_i a_{ii}^\infty \pi_j \right) \]

where \((a_{ij}^\infty)_{1 \leq i \leq n}\) is the \((i,i)\) element of

\[ a^\infty := \lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} (a_{ij}(t))_{1 \leq i,j \leq n} dt = \sum_{p \in \Sigma} \theta_p s_p s_p' \cdot \]

Asymptotic target portfolio maximizes the excess growth \(\gamma_\pi^\infty:\)

\[ \bar{\pi} := \arg \max_{\pi \in \Delta^n_+} \left( \sum_{i=1}^{n} \pi_i a_{ii}^\infty - \sum_{i=1}^{n} \pi_i a_{ii}^\infty \pi_j \right). \]

We obtain

\[ \bar{\pi}_i = \frac{1}{2} \left[ 1 - \frac{n-2}{a_{ii}^\infty} \left( \sum_{j=1}^{n} \frac{1}{a_{jj}^\infty} \right)^{-1} \right] = \lim_{t \to \infty} \Pi_i^*(t); \quad i = 1, \ldots, n. \]
Asymptotic Target Portfolio

Under the hybrid Atlas model with the assumptions

\[ v(\pi) := \lim_{T \to \infty} \frac{1}{T} \log V^\pi(T) = \gamma + \frac{1}{2} \left( \sum_{i=1}^{n} \pi_i a_{ii}^\infty - \sum_{i=1}^{n} \pi_i a_{ii}^\infty \pi_j \right) \]

where \((a_{ij}^\infty)_{1 \leq i \leq n}\) is the \((i,i)\) element of

\[ a^\infty := \lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} (a_{ij}(t))_{1 \leq i,j \leq n} d t = \sum_{p \in \Sigma_n} \theta_{p^5} p_{5'}^p. \]

Asymptotic target portfolio maximizes the excess growth \(\gamma^\infty_\pi\) :

\[ \bar{\pi} := \arg \max_{\pi \in \Delta^n_+} \left( \sum_{i=1}^{n} \pi_i a_{ii}^\infty - \sum_{i=1}^{n} \pi_i a_{ii}^\infty \pi_j \right). \]

We obtain

\[ \bar{\pi}_i = \frac{1}{2} \left[ 1 - \frac{n-2}{a_{ii}^\infty} \left( \sum_{j=1}^{n} \frac{1}{a_{jj}^\infty} \right)^{-1} \right] = \lim_{t \to \infty} \Pi^*_i(t); \quad i = 1, \ldots, n. \]
Universal Portfolio (Cover (’91) & Jamshidian (’92))

Universal portfolio is defined as

$$\hat{\Pi}_i(\cdot) := \frac{\int_{\Delta^+} \pi_i \mathcal{V}(\cdot) d\pi}{\int_{\Delta^+} \mathcal{V}(\cdot) d\pi}, \quad 1 \leq i \leq n, \quad \hat{\mathcal{V}}(\cdot) = \frac{\int_{\Delta^+} \mathcal{V}(\cdot) d\pi}{\int_{\Delta^+} d\pi}. $$

Proposition: Under the hybrid Atlas model with the model assumptions,

$$\lim_{T \to \infty} \frac{1}{T} \log \frac{\hat{\mathcal{V}}(T)}{\hat{\mathcal{V}}(\cdot)} = \lim_{T \to \infty} \frac{1}{T} \log \frac{\hat{\mathcal{V}}(T)}{\mathcal{V}^*(T)} = 0 \quad \mathbb{P} - a.s.$$


Universal Portfolio (Cover (’91) & Jamshidian (’92))

Universal portfolio is defined as

\[ \hat{\Pi}_i(\cdot) := \frac{\int_{\Delta^n_+} \pi_i \, V^\pi(\cdot) \, d\pi}{\int_{\Delta^n_+} V^\pi(\cdot) \, d\pi}, \quad 1 \leq i \leq n, \quad V^{\hat{\Pi}}(\cdot) = \frac{\int_{\Delta^n_+} V^\pi(\cdot) \, d\pi}{\int_{\Delta^n_+} d\pi}. \]

**Proposition** Under the hybrid Atlas model with the model assumptions,

\[ \lim_{T \to \infty} \frac{1}{T} \log \frac{V^{\hat{\Pi}}(T)}{V^\pi(T)} = \lim_{T \to \infty} \frac{1}{T} \log \frac{V^{\hat{\Pi}}(T)}{V^*_T(T)} = 0 \quad \mathbb{P} - a.s. \]
Conclusion

- Ergodic properties of Hybrid Atlas model
- Diversity weighted portfolio, Target portfolio, Universal portfolio.
- Further topics: short term arbitrage, generalized portfolio generating function, large market \((n \to \infty)\), numéraire portfolio, data implementation.

References:

1. arXiv: 0909.0065

Tomoyuki Ichiba (UCSB)
ichiba@pstat.ucsb.edu