Hybrid Atlas Model

of financial equity market

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November 2009

Outline

Flow of Capital

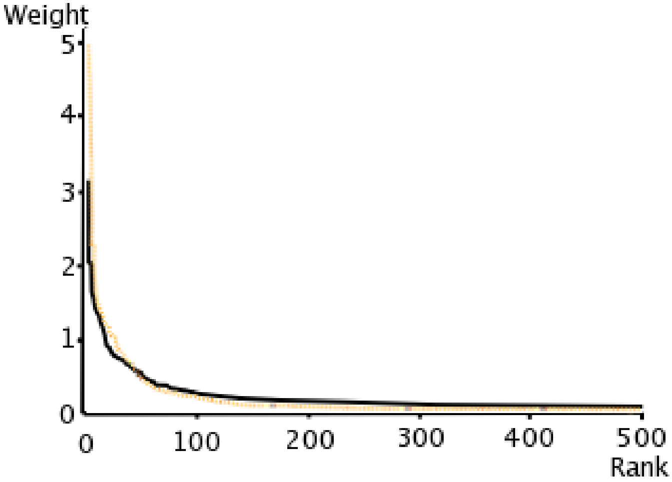

Figure: Capital Distribution Curves (Percentage) for the S&P 500 Index of 1997 (Solid Line) and 1999 (Broken Line).

Log-Log Capital Distribution Curves

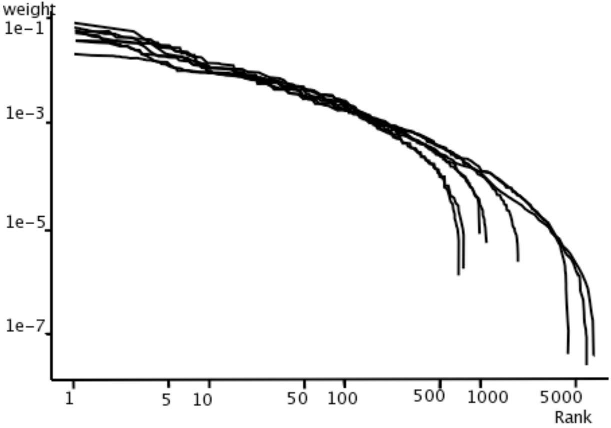

Figure: Capital distribution curves for 1929 (shortest curve) - 1999 (longest curve), every ten years. Source Fernholz('02).

What kind of models can describe this long-term stability?

A Model of Rankings [Hybrid Atlas model]

- Capital process $X := \{(X_1(t), \ldots, X_n(t)), \;\; 0 \le t < \infty\}$.
- Order Statistics:

$$X_{(1)}(t) \ge \cdots \ge X_{(n)}(t)\,; \quad 0 \le t < \infty\,.$$

Log capital $Y := \log X$:

$$Y_{(1)}(t) \ge \cdots \ge Y_{(n)}(t)\,; \quad 0 \le t < \infty\,.$$

Dynamics of log capital:

$$d\,Y_{(k)}(t) = (\gamma + \gamma_i + g_k)\,d\,t + \sigma_k\,d\,W_i(t) \quad \text{if} \;\; Y_{(k)}(t) = Y_i(t)\,;$$

for $1 \le i\,, k \le n\,,\; 0 \le t < \infty\,$, where $W(\cdot)$ is n–dim. B. M.

	company name i	kth ranked company $*$
Drift ("mean") Diffusion ("variance")	γ_i	g_k $\sigma_k > 0$

$*$ Banner, Fernholz & Karatzas ('05), Chatterjee & Pal ('07, '09), Pal & Pitman ('08).

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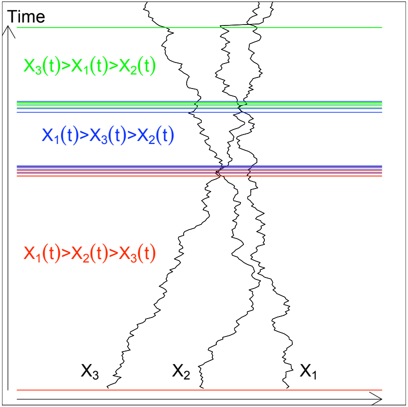
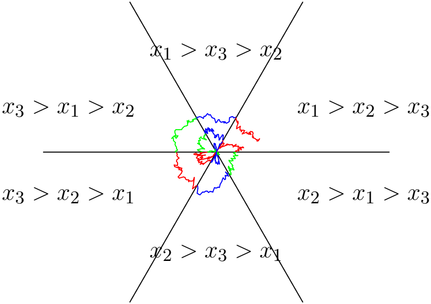
Illustration ($n = 3$) of interactions through rank





Paths in $\mathbb{R}_+ \times$ Time . A path in different wedges of $\mathbb{R}^n$.

Symmetric group Σ_n of permutations of $\{1, \ldots, n\}$.
For $n = 3$,
$\{(1,2,3),(1,3,2),(2,1,3),(2,3,1),(3,1,2),(3,2,1)\}$.

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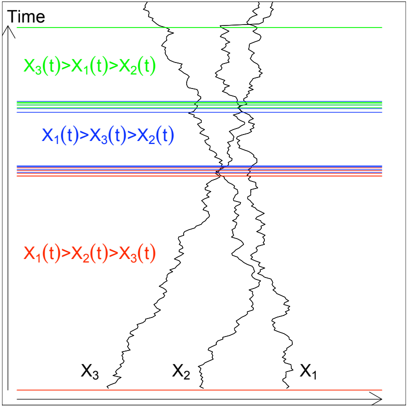

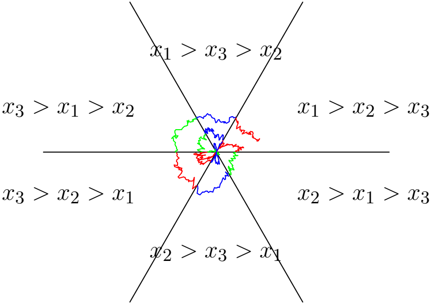

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Vector Representation

$$d\,Y(t) = G(Y(t))d\,t + S(Y(t))dW(t)\,; \quad 0 \le t < \infty$$

Σ_n : symmetric group of permutations of $\{1, 2, \ldots, n\}$.

For $\mathbf{p} \in \Sigma_n$ define wedges (chambers)

$$\mathcal{R}_{\mathbf{p}} := \{x \in \mathbb{R}^n : x_{\mathbf{p}(1)} \ge x_{\mathbf{p}(2)} \ge \cdots \ge x_{\mathbf{p}(n)}\}, \quad \mathbb{R}^n = \cup_{\mathbf{p}\in\Sigma_n} \mathcal{R}_{\mathbf{p}}\,,$$

(the inner points of $\mathcal{R}_{\mathbf{p}}$ and $\mathcal{R}_{\mathbf{p}'}$ are disjoint for $\mathbf{p} \neq \mathbf{p}' \in \Sigma_n$),

$$Q_k^{(i)} := \{x \in \mathbb{R}^n : x_i \text{ is ranked } k\text{th among } (x_1, \ldots, x_n)\}$$
$$= \cup_{\{\mathbf{p}\,:\,\mathbf{p}(k)=i\}} \mathcal{R}_{\mathbf{p}}\,; \quad 1 \le i, k \le n\,,$$

$$\cup_{j=1}^n Q_k^{(j)} = \mathbb{R}^n = \cup_{\ell=1}^n Q_\ell^{(i)} \text{ and } \mathcal{R}_{\mathbf{p}} = \cap_{k=1}^n Q_k^{(\mathbf{p}(k))}\,.$$

$$G(y) = \sum_{\mathbf{p}\in\Sigma_n} (g_{\mathbf{p}^{-1}(1)} + \gamma_1 + \gamma, \ldots, g_{\mathbf{p}^{-1}(n)} + \gamma_n + \gamma)' \cdot \mathbf{1}_{\mathcal{R}_{\mathbf{p}}}(y)\,,$$
$$S(y) = \sum_{\mathbf{p}\in\Sigma_n} \underbrace{\mathrm{diag}(\sigma_{\mathbf{p}^{-1}(1)}, \ldots, \sigma_{\mathbf{p}^{-1}(n)})}_{\mathfrak{S}_{\mathbf{p}}} \cdot \mathbf{1}_{\mathcal{R}_{\mathbf{p}}}(y)\,; \quad y \in \mathbb{R}^n\,.$$

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Martingale Problem

Theorem [Krylov('71), Stroock & Varadhan('79) , Bass & Pardoux('87)] Suppose that the coefficients $G(\cdot)$ and $a(\cdot) := SS'(\cdot)$ are bounded and measurable, and that $a(\cdot)$ is uniformly positive-definite and piecewise constant in each wedge. For each $y_0 \in \mathbb{R}^n$ there is a unique one probability measure $\mathbb{P}$ on $C([0,\infty),\mathbb{R}^n)$ such that $\mathbb{P}(Y_0 = y_0) = 1$ and

$$f(Y_t) - f(Y_0) - \int_0^t L\, f(Y_s)\, d\, s\, ; \quad 0 \le t < \infty$$

is a $\mathbb{P}$ local martingale for every $f \in C^2(\mathbb{R}^2)$ where

$$L\, f(x) = \frac{1}{2} \sum_{i,j=1}^n a_{ij}(x) D_{ij} f(x) + \sum_{i=1}^n G_i(x) D_i f(x)\, ; \;\; x \in \mathbb{R}^n.$$

This implies that the hybrid Atlas model is well-defined.

Model

Market capitalization X follows Hybrid Atlas model: the log capitalization $Y_i = \log X_i$ of company i has

drift $\gamma + g_k + \gamma_i$ and volatility σ_k ,

when company i is k^{th}ranked, i.e., $Y \in Q_k^{(i)}$ for $1 \le k, i \le n$.

$$d\,Y_i(t) = \Big(\gamma + \sum_{k=1}^{n} g_k \mathbf{1}_{Q_k^{(i)}}(Y(t)) + \gamma_i\Big) d\,t + \sum_{k=1}^{n} \sigma_k \mathbf{1}_{Q_k^{(i)}}(Y(t)) d\,W_i(t)\,; \quad 0 \le t < \infty\,.$$

Model assumptions

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Assume $\sigma_k > 0$, $(g_k, 1 \le k \le n)$, $(\gamma_i, 1 \le i \le n)$ and γ are real constants with stability conditions

$$\sum_{k=1}^{n} g_k + \sum_{i=1}^{n} \gamma_i = 0 \,, \quad \sum_{\ell=1}^{k} (g_\ell + \gamma_{\mathbf{p}(\ell)}) < 0 \,, \quad k = 1, \ldots, n-1, \ \mathbf{p} \in \Sigma_n \,.$$

- $\gamma_i = 0, 1 \le i \le n, \quad g_1 = \cdots = g_{n-1} = -g < 0,$
  $g_n = (n-1)g > 0 \,.$
- $\gamma_i = 1 - (2i)/(n+1), 1 \le i \le n,$
  $g_k = -1, k = 1, \ldots, n-1, \ g_n = n - 1 \,.$

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Model Summary

The log-capitalization $Y = \log X$ follows

$$d\,Y_i(t) = \Big(\gamma + \sum_{k=1}^{n} g_k \mathbf{1}_{Q_k^{(i)}}(Y(t)) + \gamma_i\Big) d\,t + \sum_{k=1}^{n} \sigma_k \mathbf{1}_{Q_k^{(i)}}(Y(t)) d\,W_i(t)\,; \quad 0 \le t < \infty$$

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Stochastic stability

The average $\overline{Y}(\cdot) := \sum_{i=1}^{n} Y_i(\cdot) \,/\, n$ of log-capitalization:

$$d\overline{Y}(t) = \gamma\, d\,t + \frac{1}{n}\sum_{k=1}^{n} \sigma_k \underbrace{\sum_{i=1}^{n} \mathbf{1}_{Q_k^{(i)}}(Y(t))d\,W_i(t)}_{d\,B_k(t)}$$

is a Brownian motion with variance rate $\sum_{k=1}^{n} \sigma_k^2/n^2$ drift γ by the Dambis-Dubins-Schwartz Theorem.

Proposition Under the assumptions the deviations $\widetilde{Y}(\cdot) := (Y_1(\cdot) - \overline{Y}(\cdot), \ldots, Y_n(\cdot) - \overline{Y}(\cdot))$ from the average are stable in distribution, i.e., there is a unique invariant probability measure $\mu(\cdot)$ such that for every bounded, measurable function f we have the Strong Law of Large Numbers

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T f(\widetilde{Y}(t))\,d\,t = \int_\Pi f(y)\mu(d\,y)\,, \quad a.s.$$

where $\Pi := \{y \in \mathbb{R}^n : y_1 + \cdots + y_n = 0\}$.

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where $\Pi := \{ y \in \mathbb{R}^n : y_1 + \cdots + y_n = 0 \}$.

Average occupation times

Especially taking $f(\cdot) = \mathbf{1}_{\mathcal{R}_\mathbf{p}}(\cdot)$ or $\mathbf{1}_{Q_k^{(i)}}(\cdot)$, we define from μ the average occupation time of X in $\mathcal{R}_\mathbf{p}$ or $Q_k^{(i)}$:

$$\theta_\mathbf{p} := \mu(\mathcal{R}_\mathbf{p}) = \lim_{T\to\infty} \frac{1}{T}\int_0^T \mathbf{1}_{\mathcal{R}_\mathbf{p}}(X(t))dt$$

$$\theta_{k,i} := \mu(Q_k^{(i)}) = \lim_{T\to\infty} \frac{1}{T}\int_0^T \mathbf{1}_{Q_k^{(i)}}(X(t))dt, \quad 1 \le k, i \le n,$$

since $\mathbf{1}_{\mathcal{R}_\mathbf{p}}(\widetilde{Y}(\cdot)) = \mathbf{1}_{\mathcal{R}_\mathbf{p}}(X(\cdot))$ and $\mathbf{1}_{Q_k^{(i)}}(X(\cdot)) = \mathbf{1}_{Q_k^{(i)}}(\widetilde{Y}(\cdot))$. By definition

- $0 \le \theta_{k,i} = \sum_{\{\mathbf{p}\in\Sigma_n:\mathbf{p}(k)=i\}} \theta_\mathbf{p} \le 1$ for $1 \le k, i \le n$,
- $\sum_{\ell=1}^n \theta_{\ell,i} = \sum_{j=1}^n \theta_{k,j} = 1$ for $1 \le k, i \le n$.

What is the invariant distribution μ?

Attainability

- One-dimensional Brownian motion attains the origin infinitely often.
- Two-dimensional Brownian motion does not attain the origin.

Does the process $X(\cdot)$ attain the origin?

$$X(t) = X(0) + \int_0^t b(X(s))\,ds + \int_0^t \sigma(X(s))dW(s)$$

where b and σ are bounded measurable functions.

- Friedman('74), Bass & Pardoux('87).

Effective Dimension

Let us define *effective dimension* $\mathrm{ED}(\cdot)$ by

$$\mathrm{ED}(x) = \frac{\mathrm{trace}(A(x))\|x\|^2}{x'A(x)x}\,; \quad x \in \mathbb{R}^n \setminus \{0\}\,,$$

where $A(\cdot) = \sigma(\cdot)\sigma(\cdot)'$.

Proposition

Suppose $X(0) \neq 0$.
If $\inf_{x\in\mathbb{R}^n\setminus\{0\}} ED(x) \geq 2$, *then* $X(\cdot)$ *does not attain the origin.*
If $\sup_{x\in\mathbb{R}^n\setminus\{0\}} ED(x) < 2$ *and if there is no drift, i.e.,* $b(\cdot) \equiv 0$, *then* $X(\cdot)$ *attains the origin.*

- Exterior Dirichlet Problem by Meyers and Serrin('60).
- Removal of drift by Girsanov's theorem.
- If there is drift, take $\frac{[\mathrm{trace(A(x))}+x'b(x)]\cdot\|x\|^2}{x'A(x)x}$.

Triple collision

Now consider *triple collision*:

$$\{X_i(t) = X_j(t) = X_k(t) \quad \text{for some } t > 0\,, \quad 1 \le i < j < k \le n\}\,.$$

What is the probability of triple collision?

Fix $i = 1$, $j = 2$, $k = 3$. Let us define the sum of squared distances:

$$s^2(x) := (x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_3 - x_1)^2 \; = x' DD' x\,; \quad x \in \mathbb{R}^n\,,$$

where $(n \times 3)$ matrix D is defined by $D := (d_1, d_2, d_3)$ with

$$d_1 := (1, -1, 0, \ldots, 0)'\,, \quad d_2 := (0, 1, -1, 0, \ldots, 0)'\,,$$
$$d_3 := (-1, 0, 1, \ldots, 0)'\,.$$

$$\mathcal{Z} := \{x \in \mathbb{R}^n : s(x) = 0\}.$$

Define the local effective dimension:

$$R(x) := \frac{\mathrm{trace}(D'A(x)D)\ x'DD'x}{x'DD'A(x)DD'x}\,; \quad x \in \mathbb{R}^n \setminus \mathcal{Z}\,.$$

Proposition

Suppose $s(X(0)) \neq 0$. *If* $\inf_{x\in\mathbb{R}^n\setminus\mathcal{Z}} R(x) \geq 2$, *then*

$$\mathbb{P}(X_1(t) = X_2(t) = X_3(t) \text{ for some } t \geq 0) = 0\,.$$

If $\sup_{x\in\mathbb{R}^n\setminus\mathcal{Z}} R(x) < 2$ *and if there is no drift, i.e.,* $b(\cdot) \equiv 0$, *then*

$$\mathbb{P}(X_1(t) = X_2(t) = X_3(t) \text{ for some } t \geq 0) = 1\,.$$

- $R(\cdot) \equiv 2$ for $n-$dim. BM, i.e., $A(\cdot) \equiv I$.
- If there is drift, take $\frac{[\mathrm{trace}(D'A(x)D)+x'DD'b(x)]\cdot x'DD'x}{x'DD'A(x)DD'x}$.

Idea of Proof: a comparison with Bessel process with dimension two.

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Idea of Proof: a comparison with Bessel process with dimension two.

Rankings

Recall $Y_{(1)}(\cdot) \geq Y_{(2)}(\cdot) \geq \cdots \geq Y_{(n)}(\cdot)$. Let us denote by $\Lambda^{k,j}(t)$ the local time accumulated at the origin by the nonnegative semimartingale $Y_{(k)}(\cdot) - Y_{(j)}(\cdot)$ up to time t for $1 \leq k < j \leq n$.

Theorem[Banner & Ghomrasni ('07)] For a general class of semimartingale $Y(\cdot)$, the rankings satisfy

$$d\,Y_{(k)}(t) = \sum_{i=1}^{n} \mathbf{1}_{Q_k^{(i)}}(Y(t))d\,Y_i(t)$$

$$+ (N_k(t))^{-1}\Big[\sum_{\ell=k+1}^{n} d\,\Lambda^{k,\ell}(t) - \sum_{\ell=1}^{k-1} d\,\Lambda^{\ell,k}(t)\Big]$$

where $N_k(t)$ is the cardinality $|\{i : Y_i(t) = Y_{(k)}(t)\}|$.

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Recall $Y_{(1)}(\cdot) \geq Y_{(2)}(\cdot) \geq \cdots \geq Y_{(n)}(\cdot)$. Let us denote by $\Lambda^{k,j}(t)$ the local time accumulated at the origin by the nonnegative semimartingale $Y_{(k)}(\cdot) - Y_{(j)}(\cdot)$ up to time t for $1 \leq k < j \leq n$.

Lemma Under the non-degeneracy condition $\sigma_k > 0$ for $k = 1, \ldots, n$,

$$dY_{(k)}(t) = \Big(\gamma + g_k + \sum_{i=1}^{n} \gamma_i 1_{Q_k^{(i)}}(Y(t))\Big)\, d\,t + \sigma_k\, d\, B_k(t) + \frac{1}{2}\Big[d\,\Lambda^{k,k+1}(t) - d\,\Lambda^{k-1,k}(t)\Big].$$

for $k = 1, \ldots, n$, $0 \leq t \leq T$.

Idea of Proof: a comparison with a Bessel process with dimension one to show $\Lambda^{k,\ell}(\cdot) \equiv 0$, $|k - \ell| \geq 2$.

Rankings

Recall $Y_{(1)}(\cdot) \geq Y_{(2)}(\cdot) \geq \cdots \geq Y_{(n)}(\cdot)$. Let us denote by $\Lambda^{k,j}(t)$ the local time accumulated at the origin by the nonnegative semimartingale $Y_{(k)}(\cdot) - Y_{(j)}(\cdot)$ up to time t for $1 \leq k < j \leq n$.

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$$dY_{(k)}(t) = \Big(\gamma + g_k + \sum_{i=1}^{n} \gamma_i 1_{Q_k^{(i)}}(Y(t))\Big)\, dt + \sigma_k \, dB_k(t) + \frac{1}{2}\Big[d\Lambda^{k,k+1}(t) - d\Lambda^{k-1,k}(t)\Big].$$

for $k = 1, \ldots, n$, $0 \leq t \leq T$.

Idea of Proof: a comparison with a Bessel process with dimension one to show $\Lambda^{k,\ell}(\cdot) \equiv 0$, $|k - \ell| \geq 2$.

Long-term growth relations

Proposition Under the assumptions we obtain the following long-term growth relations:

$$\lim_{T\to\infty} \frac{Y_i(T)}{T} = \lim_{T\to\infty} \frac{\log X_i(T)}{T} = \gamma$$
$$= \lim_{T\to\infty} \frac{\log \sum_{i=1}^{n} X_i(T)}{T} \quad a.s.$$

Thus the model is coherent:

$$\lim_{T\to\infty} \frac{1}{T} \log \mu_i(T) = 0 \quad a.s.; \quad i = 1, \dots, n$$

where $\mu_i(\cdot) = X_i(\cdot)/(X_1(\cdot) + \cdots + X_n(\cdot))$. Moreover,

$$\sum_{k=1}^{n} g_k \theta_{k,i} + \gamma_i = 0\,; \quad i = 1, \dots, n\,.$$

$$\sum_{k=1}^{n} g_k \theta_{k,i} + \gamma_i = 0 \, ; \quad i = 1, \ldots, n \, .$$

The log-capitalization Y grows with rate γ and follows

$$d\,Y_i(t) = \Big(\gamma + \sum_{k=1}^{n} g_k \mathbf{1}_{Q_k^{(i)}}(Y(t)) + \gamma_i \Big) d\,t + \sum_{k=1}^{n} \sigma_k \mathbf{1}_{Q_k^{(i)}}(Y(t)) d\,W_i(t) \, ; \quad 0 \le t < \infty$$

for $i = 1, \ldots, n$. The ranking $(Y_{(1)}(\cdot), \ldots, Y_{(n)}(\cdot))$ follows

$$dY_{(k)}(t) = \Big(\gamma + g_k + \sum_{i=1}^{n} \gamma_i \mathbf{1}_{Q_k^{(i)}}(Y(t)) \Big) \, d\,t + \sigma_k \, d\,B_k(t) + \frac{1}{2} \Big[d\,\Lambda^{k,k+1}(t) - d\,\Lambda^{k-1,k}(t) \Big] \, .$$

for $k = 1, \ldots, n$, $0 \le t < \infty$.

Semimartingale reflected Brownian motions

The adjacent differences (gaps) $\Xi(\cdot) := (\Xi_1(\cdot), \ldots, \Xi_n(\cdot))'$ where $\Xi_k(\cdot) := Y_{(k)}(\cdot) - Y_{(k+1)}(\cdot)$ for $k = 1, \ldots, n-1$ can be seen as a semimartingale reflected Brownian motion (SRBM):

$$\Xi(t) = \Xi(0) + \zeta(t) + (I_n - \mathfrak{Q})\Lambda(t)$$

where $\zeta(\cdot) := (\zeta_1(\cdot), \ldots, \zeta_n(\cdot))'$, $\Lambda(\cdot) := (\Lambda^{1,2}(\cdot), \ldots, \Lambda^{n-1,n}(\cdot))'$,

$$\zeta_k(\cdot) := \sum_{i=1}^{n} \int_0^{\cdot} \mathbf{1}_{Q_k^{(i)}}(Y(s))\, d\, Y(s) - \sum_{i=1}^{n} \int_0^{\cdot} \mathbf{1}_{Q_{k+1}^{(i)}}(Y(s))\, d\, Y(s)$$

for $k = 1, \ldots, n-1$, and $\mathfrak{Q}$ is an $(n-1) \times (n-1)$ matrix with elements

$$\mathfrak{Q} := \begin{pmatrix} 0 & 1/2 & & & & \\ 1/2 & 0 & 1/2 & & & \\ & 1/2 & \ddots & \ddots & & \\ & & \ddots & 0 & 1/2 \\ & & & 1/2 & 0 \end{pmatrix} .$$

Thus the gaps $\Xi_k := Y_{(k)}(\cdot) - Y_{(k+1)}(\cdot)$ follow

$$\Xi(t) = \Xi(0) + \underbrace{\zeta(t)}_{\text{semimartingale}} + \underbrace{(I_n - \mathfrak{Q})\Lambda(t)}_{\text{reflection part}}$$

In order to study the invariant measure μ, we apply the theory of semimartingale reflected Brownian motions developed by M. Harrison, M. Reiman, R. Williams and others.

In addition to the model assumptions, we assume linearly growing variances:

$$\sigma_2^2 - \sigma_1^2 = \sigma_3^2 - \sigma_2^2 = \cdots = \sigma_n^2 - \sigma_{n-1}^2 .$$

Invariant distribution of gaps and index

Let us define the indicator map $\mathbb{R}^n \ni x \mapsto \mathfrak{p}^x \in \Sigma_n$ such that $x_{\mathfrak{p}^x(1)} \ge x_{\mathfrak{p}^x(2)} \ge \cdots \ge x_{\mathfrak{p}(n)}$, and the index process $\mathfrak{P}_t := \mathfrak{p}^{Y(t)}$.

Proposition Under the stability and the linearly growing variance conditions the invariant distribution $\nu(\cdot)$ of $(\Xi(\cdot), \mathfrak{P}_\cdot)$ is

$$\nu(A \times B) = \Big(\sum_{\mathbf{q} \in \Sigma_n} \prod_{k=1}^{n-1} \lambda_{\mathbf{q},k}^{-1} \Big)^{-1} \sum_{\mathbf{p} \in \Sigma_n} \int_A \exp(-\langle \lambda_{\mathbf{p}}, z \rangle) d\, z$$

for every measurable set $A \times B$ where $\lambda_{\mathbf{p}} := (\lambda_{\mathbf{p},1}, \ldots, \lambda_{\mathbf{p},n-1})'$ is the vector of components

$$\lambda_{\mathbf{p},k} := \frac{-4(\sum_{\ell=1}^{k} g_\ell + \gamma_{\mathbf{p}(\ell)})}{\sigma_k^2 + \sigma_{k+1}^2} > 0\,; \quad \mathbf{p} \in \Sigma_n\,,\ 1 \le k \le n-1\,.$$

Proof: an extension from M. Harrison and R. Williams ('87).

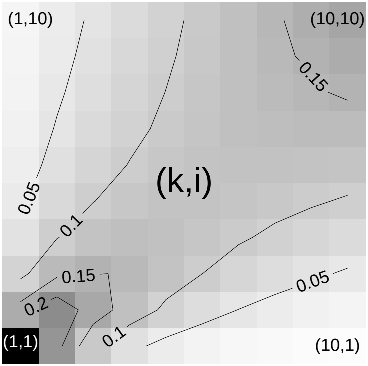
Average occupation time

Corollary The average occupation times are

$$\theta_{\mathbf{p}} = \Big(\sum_{\mathbf{q} \in \Sigma_n} \prod_{k=1}^{n-1} \lambda_{\mathbf{q},k}^{-1} \Big)^{-1} \prod_{j=1}^{n-1} \lambda_{\mathbf{p},j}^{-1} \quad \text{and} \quad \theta_{k,i} = \sum_{\{\mathbf{p} \in \Sigma_n : \mathbf{p}(k) = i\}} \theta_{\mathbf{p}}$$

for $\mathbf{p} \in \Sigma_n$ and $1 \leq k, i \leq n$.

- If all $\gamma_i = 0$ and $\sigma_1^2 = \cdots \sigma_n^2$, then $\theta_{k,i} = \frac{1}{n}$ for $1 \leq k, i \leq n$.
- Heat map of $\theta_{k,i}$ when $n = 10$, $\sigma_k^2 = 1 + k$, $g_k = -1$ for $k = 1, \ldots, 9$, $g_{10} = 9$, and $\gamma_i = 1 - (2i)/(n+1)$ for $i = 1, \ldots, n$.



Market weights come from Pareto type

Corollary The joint invariant distribution of market shares

$$\mu_{(i)}(\cdot) := X_{(i)}(\cdot)/(X_1(\cdot) + \cdots + X_n(\cdot))\,; \quad i = 1, \ldots, n$$

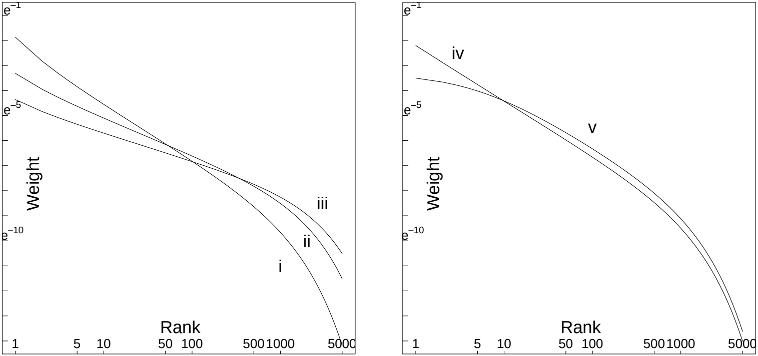
has the density

$$\wp(m_1, \ldots, m_{n-1})$$
$$= \sum_{\mathbf{p}\in\Pi_n} \theta_{\mathbf{p}} \frac{\lambda_{\mathbf{p},1}\cdots\lambda_{\mathbf{p},n-1}}{m_1^{\lambda_{\mathbf{p},1}+1}\cdot m_2^{\lambda_{\mathbf{p},2}-\lambda_{\mathbf{p},1}+1}\cdots m_{n-1}^{\lambda_{\mathbf{p},n-1}-\lambda_{\mathbf{p},n-2}+1} m_n^{-\lambda_{\mathbf{p},n-1}+1}}\,,$$
$$0 < m_n \le m_{n-1} \le \ldots \le m_1 < 1,$$
$$m_n = 1 - m_1 - \cdots - m_{n-1}\,.$$

This is a distribution of ratios of Pareto type disribution.

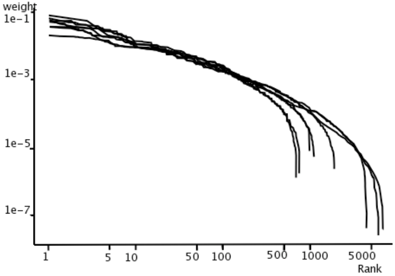
Expected capital distribution curves

From the expected slopes $\mathbb{E}^{\nu}[\frac{\log \mu_{(k)} - \log \mu_{(k-1)}}{\log(k+1) - \log k}] = -\frac{\mathbb{E}^{\nu}(\Xi_k)}{log(1+k^{-1})}$ we obtain expected capital distribution curves.



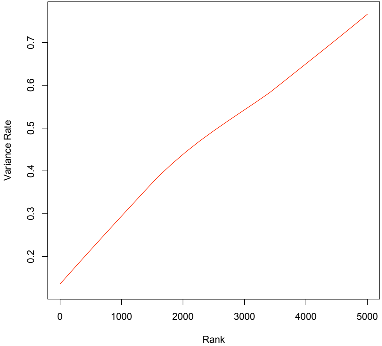
- $n = 5000$, $g_n = c_*(2n-1)$, $g_k = 0$, $1 \le k \le n-1$, $\gamma_1 = -c_*$, $\gamma_i = -2c_*$, $2 \le i \le n$, $\sigma_k^2 = 0.075 + 6k \times 10^{-5}$, $1 \le k \le n$. (i) $c_* = 0.02$, (ii) $c_* = 0.03$, (iii) $c_* = 0.04$.
- (iv) $c_* = 0.02$, $g_1 = -0.016$, $g_k = 0$, $2 \le k \le n-1$, $g_n = (0.02)(2n-1) + 0.016$,
- (v) $g_1 = \cdots = g_{50} = -0.016$, $g_k = 0$, $51 \le k \le n-1$, $g_n = (0.02)(2n-1) + 0.8$.

Empirical data



Historical capital distribution curves. Data 1929-1999.

Source: Fernholz ('02)



Growing variances. 1990-1999.

Capital Stocks and Portfolio Rules

- Market $X = ((X_1(t), \ldots, X_n(t)), t \geq 0)$ of n companies

$$\log \frac{X_i(T)}{X_i(0)} = \int_0^T G_i(t)dt + \int_0^T \sum_{\nu=1}^n S_{i,\nu}(t)dW_\nu(t),$$

with initial capital $X_i(0) = x_i > 0,\ i = 1, \ldots, n,\ 0 \leq T < \infty$.
Define $a_{ij}(\cdot) = \sum_{\nu=1}^n S_{i\nu}(\cdot)S_{j\nu}(\cdot)$ and $A_{ij}(\cdot) = \int_0^\cdot a_{ij}(t)d\,t$.

- Long only Portfolio rule π and its wealth V^π.
Choose $\pi \in \Delta_+^n := \{x \in \mathbb{R}^n : \sum x_i = 1,\ x_i \geq 0\}$
invest $\pi_i V^\pi$ of money to company i for $i = 1, \ldots, n$, i.e.,
$\pi_i V^\pi / X_i$ share of company i:

$$dV^\pi(t) = \sum_{i=1}^n \frac{\pi_i(t)V^\pi(t)}{X_i(t)} dX_i(t), \quad 0 \leq t < \infty,$$

$$V^\pi(0) = w.$$

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Portfolios and Relative Arbitrage

- Market portfolio: Take
$\pi(t) = \mathfrak{m}(t) = (\mathfrak{m}_1(t), \cdots, \mathfrak{m}_n(t)) \in \Delta^n_+$ where

$$\mathfrak{m}_i(t) = \frac{X_i(t)}{X_1 + \cdots X_n(t)}, \quad i = 1, \ldots, n, \quad 0 \le t < \infty .$$

- Diversity weighted portfolio: Given $p \in [0, 1]$, take
$\pi_i(t) = \frac{(\mathfrak{m}_i(t))^p}{\sum_{j=1}^n (\mathfrak{m}_j(t))^p}$ for $i = 1, \ldots, n,\ 0 \le t < \infty$.
- Functionally generated portfolio (Fernholz ('02) & Karatzas ('08)).
- A portfolio π represents an arbitrage opportunity relative to another portfolio ρ on $[0, T]$, if

$$\mathbb{P}(V^{\pi}(T) \ge V^{\rho}(T)) = 1, \quad \mathbb{P}(V^{\pi}(T) > V^{\rho}(T)) > 0.$$

Can we find an arbitrage opportunity π relative to $\mathfrak{m}$?

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Can we find an arbitrage opportunity π relative to $\mathfrak{m}$?

Constant-portfolio

For a constant-proportion $\pi(\cdot) \equiv \pi$,

$$V^{\pi}(t) = w \cdot \exp\Big[\sum_{i=1}^{n} \pi_i \cdot \Big\{ \frac{A_{ii}(t)}{2} + \log\Big(\frac{X_i(t)}{X_i(0)} \Big) \Big\} - \frac{1}{2} \sum_{i,j=1}^{n} \pi_i A_{ij}(t) \pi_j \Big]$$

for $0 \le t < \infty$.

Here $A_{ij}(\cdot) = \int_0^{\cdot} a_{ij}(t) d\, t$ and $a_{ij}(\cdot) = \sum_{\nu=1}^{n} S_{i\nu}(\cdot) S_{j\nu}(\cdot)$,

$$d\, Y(t) = G(Y(t)) d\, t + S(Y(t)) dW(t) \,; \quad 0 \le t < \infty \,,$$

$$G(y) = \sum_{\mathbf{p} \in \Sigma_n} (g_{\mathbf{p}^{-1}(1)} + \gamma_1 + \gamma, \ldots, g_{\mathbf{p}^{-1}(n)} + \gamma_n + \gamma)' \cdot \mathbf{1}_{\mathcal{R}_{\mathbf{p}}}(y) \,,$$
$$S(y) = \sum_{\mathbf{p} \in \Sigma_n} \underbrace{\mathrm{diag}(\sigma_{\mathbf{p}^{-1}(1)}, \ldots, \sigma_{\mathbf{p}^{-1}(n)})}_{\mathfrak{s}_{\mathbf{p}}} \cdot \mathbf{1}_{\mathcal{R}_{\mathbf{p}}}(y) \,; \quad y \in \mathbb{R}^n .$$

Target Portfolio(Cover('91) & Jamshidian('92))

$$V^{\pi}(\cdot) = w \cdot \exp\Big[\sum_{i=1}^{n} \pi_i \Big\{ \frac{A_{ii}(t)}{2} + \log\Big(\frac{X_i(\cdot)}{X_i(0)}\Big)\Big\} - \frac{1}{2}\sum_{i,j=1}^{n} \pi_i A_{ij}(\cdot)\pi_j\Big]$$

Target Portfolio $\Pi^*(t)$ maximizes the wealth $V^{\pi}(t)$ for $t \geq 0$:

$$V_*(t) := \max_{\pi \in \Delta_+^n} V^{\pi}(t), \quad \Pi^*(t) := \arg\max_{\pi \in \Delta_+^n} V^{\pi}(t),$$

where by Lagrange method we obtain

$$\Pi_i^*(t) = \Big(2A_{ii}(t)\sum_{j=1}^{n}\frac{1}{A_{jj}(t)}\Big)^{-1}\Big[2 - n - 2\sum_{j=1}^{n}\frac{1}{A_{jj}(t)}\log\Big(\frac{X_j(t)}{X_j(0)}\Big)\Big] + \frac{1}{2} + \frac{1}{A_{ii}(t)}\log\Big(\frac{X_i(t)}{X_i(0)}\Big); \quad 0 \leq t < \infty.$$

Asymptotic Target Portfolio

Under the hybrid Atlas model with the assumptions

$$v(\pi) := \lim_{T\to\infty} \frac{1}{T} \log V^{\pi}(T) = \gamma + \underbrace{\frac{1}{2}\Big(\sum_{i=1}^{n} \pi_i \mathfrak{a}_{ii}^{\infty} - \sum_{i=1}^{n} \pi_i \mathfrak{a}_{ii}^{\infty} \pi_j\Big)}_{\gamma_{\pi}^{\infty}}$$

where $(\mathfrak{a}_{ij}^{\infty})_{1\le i\le n}$ is the (i,i) element of

$$\mathfrak{a}^{\infty} := \lim_{T\to\infty} \frac{1}{T} \int_0^T (a_{ij}(t))_{1\le i,j\le n} d\,t = \sum_{\mathbf{p}\in\Sigma_n} \theta_{\mathbf{p}} \mathfrak{s}_{\mathbf{p}} \mathfrak{s}_{\mathbf{p}}' .$$

Asymptotic target portfolio maximizes the excess growth γ_{π}^{∞} :

$$\bar{\pi} := \arg\max_{\pi\in\Delta_+^n} \Big(\sum_{i=1}^{n} \pi_i \mathfrak{a}_{ii}^{\infty} - \sum_{i=1}^{n} \pi_i \mathfrak{a}_{ii}^{\infty} \pi_j\Big) .$$

We obtain

$$\bar{\pi}_i = \frac{1}{2}\Big[1 - \frac{n-2}{\mathfrak{a}_{ii}^{\infty}}\Big(\sum_{j=1}^{n} \frac{1}{\mathfrak{a}_{jj}^{\infty}}\Big)^{-1}\Big] = \lim_{t\to\infty} \Pi_i^*(t) ; \quad i = 1, \ldots, n .$$

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$$\bar{\pi}_i = \frac{1}{2}\Big[1 - \frac{n-2}{\mathfrak{a}_{ii}^{\infty}} \Big(\sum_{j=1}^{n} \frac{1}{\mathfrak{a}_{jj}^{\infty}} \Big)^{-1} \Big] = \lim_{t\to\infty} \Pi_i^{*}(t)\,; \quad i = 1, \ldots, n\,.$$

Universal Portfolio(Cover('91) & Jamshidian('92))

Universal portfolio is defined as

$$\widehat{\Pi}_i(\cdot) := \frac{\int_{\Delta_+^n} \pi_i V^\pi(\cdot) d\pi}{\int_{\Delta_+^n} V^\pi(\cdot) d\pi}, \, 1 \le i \le n, \quad V^{\widehat{\Pi}}(\cdot) = \frac{\int_{\Delta_+^n} V^\pi(\cdot) d\pi}{\int_{\Delta_+^n} d\pi}.$$

Proposition Under the hybrid Atlas model with the model assumptions,

$$\lim_{T\to\infty} \frac{1}{T} \log \frac{V^{\widehat{\Pi}}(T)}{V^{\bar{\pi}}(T)} = \lim_{T\to\infty} \frac{1}{T} \log \frac{V^{\widehat{\Pi}}(T)}{V_*(T)} = 0 \quad \mathbb{P} - a.s.$$

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$$\lim_{T \to \infty} \frac{1}{T} \log \frac{V^{\widehat{\Pi}}(T)}{V^{\bar{\pi}}(T)} = \lim_{T \to \infty} \frac{1}{T} \log \frac{V^{\widehat{\Pi}}(T)}{V_*(T)} = 0 \quad \mathbb{P} - a.s.$$

Conclusion

- Ergodic properties of Hybrid Atlas model
- Diversity weighted portfolio, Target portfolio, Universal portfolio.
- Further topics: short term arbitrage, generalized portfolio generating function, large market ($n \to \infty$), numéraire portfolio, data implementation.

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1. arXiv: 0909.0065
2. arXiv: 0810.2149 (to appear in Annals of Applied Probability)

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