A Fast Mean-Reverting Correction to Heston’s Stochastic Volatility Model

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Outline

1. Heston Model
   - Motivation and Dynamics
   - Why We like Heston
   - Problems with Heston

2. Multiscale Model
   - Motivation for Multiple Time Scales
   - Multiscale Dynamics
   - Option Pricing

3. Numerical Work
   - Multiscale Implied Volatility Surface
   - Multiscale Fit to Data
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Heston 2.0
Volatility Not Constant

Daily Log Returns

- S&P500
- Simulated GBM

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Heston 2.0
Motivated by notion that volatility not constant

\[ dX_t = rX_t dt + \sqrt{Z_t} X_t dW^X_t \]
\[ dZ_t = \kappa (\theta - Z_t) dt + \sigma \sqrt{Z_t} dW^Z_t \]
\[ d\langle W^X, W^Z \rangle_t = \rho dt \]

- One-factor stochastic volatility model
- Square of volatility, \( Z_t \), follows CIR process
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Formulas!

- Explicit formulas for European options:

\[
P_H(t, x, z) = e^{-r\tau} \frac{1}{2\pi} \int e^{-ikq} \hat{G}(\tau, k, z) \hat{h}(k) dk
\]

\[
q(t, x) = r(T - t) + \log x,
\]

\[
\hat{h}(k) = \int e^{ikq} h(e^q) dq,
\]

\[
\hat{G}(\tau, k, z) = e^{C(\tau, k) + zD(\tau, k)}
\]

\[
\ldots
\]

- \(C(\tau, k)\) and \(D(\tau, k)\) solve ODE’s in \(\tau = T - t\).
- Note: audience tunes out if you put too many equations on a slide.
Pretty Pictures!
Pretty Pictures Explained

- Implied volatility $\sigma_{imp}(T, K)$ defined by

  $$P_{BS}(\sigma_{imp}(T, K)) = P(T, K)$$

  $P$ is price of option with strike $K$ and expiration $T$

- Heston captures well-documented features of implied volatility surface: smile and skew
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Captures Some ... Not All Features of Smile

- Misprices far ITM and OTM European options [5] [12]
Simultaneous Fit Across Expirations Is Poor

- Particular difficulty fitting short expirations [7]
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What’s Wrong with Heston?

- Single factor of volatility running on single time scale not sufficient to describe dynamics of the volatility process.
- Not just Heston . . . Any one-factor stochastic volatility model has trouble fitting implied volatility levels across all strikes and maturities [7]
- Empirical evidence suggests existence of several stochastic volatility factors running on different time scales
Evidence

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Motivation for Multiple Time Scales

Multiscale Dynamics

Option Pricing

Multiscale Under Risk-Neutral Measure

\[ dX_t = rX_t dt + \sqrt{Z_t} f(Y_t) X_t dW^X_t \]
\[ dY_t = \frac{Z_t}{\epsilon} (m - Y_t) dt + \nu \sqrt{2} \frac{Z_t}{\epsilon} dW^Y_t \]
\[ dZ_t = \kappa (\theta - Z_t) dt + \sigma \sqrt{Z_t} dW^Z_t \]
\[ d\left\langle W^i, W^j \right\rangle_t = \rho_{ij} dt \hspace{1cm} i, j \in \{x, y, z\} \]

- Volatility controlled by product \( \sqrt{Z_t} f(Y_t) \)
- \( Y_t \) modeled as OU process running on time-scale \( \epsilon / Z_t \)
- Note: \( f(y) = 1 \Rightarrow \) model reduces to Heston

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Heston 2.0
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Option Pricing PDE

- Price of European Option Expressed as:

\[ P_t = \mathbb{E} \left[ e^{-r(T-t)} h(X_T) \bigg| X_t, Y_t, Z_t \right] =: P^\varepsilon(t, X_t, Y_t, Z_t) \]

- Using Feynman-Kac, derive following PDE for \( P^\varepsilon \):

\[
\mathcal{L}^\varepsilon P^\varepsilon(t, x, y, z) = 0, \\
\mathcal{L}^\varepsilon = \frac{\partial}{\partial t} + \mathcal{L}(x, y, z) - r, \\
P^\varepsilon(T, x, y, z) = h(x)
\]
Some Book-Keeping of $\mathcal{L}^\epsilon$

- $\mathcal{L}^\epsilon$ has convenient form

\[
\mathcal{L}^\epsilon = \frac{Z}{\epsilon} \mathcal{L}_0 + \frac{Z}{\sqrt{\epsilon}} \mathcal{L}_1 + \mathcal{L}_2,
\]

\[
\mathcal{L}_0 = \nu^2 \frac{\partial^2}{\partial y^2} + (m - y) \frac{\partial}{\partial y}
\]

\[
\mathcal{L}_1 = \rho_{yz} \sigma_\nu \sqrt{2} \frac{\partial^2}{\partial y \partial z} + \rho_{xy} \nu \sqrt{2} f(y) x \frac{\partial^2}{\partial x \partial y}
\]

\[
\mathcal{L}_2 = \frac{\partial}{\partial t} + \frac{1}{2} f^2(y) z x^2 \frac{\partial^2}{\partial x^2} + r \left( x \frac{\partial}{\partial x} - \cdot \right)
\]

\[
+ \frac{1}{2} \sigma^2 z \frac{\partial^2}{\partial z^2} + \kappa (\theta - z) \frac{\partial}{\partial z} + \rho_{xz} \sigma f(y) z x \frac{\partial^2}{\partial x \partial z}.
\]
Perturbative Solution

- PDE has no analytic solution for general $f(y)$
- Perform singular perturbation with respect to $\epsilon$
  \[
  P^\epsilon = P_0 + \sqrt{\epsilon} P_1 + \epsilon P_2 + \ldots
  \]
- Turns out $P_0$ and $P_1$ functions of $t$, $x$, and $z$ only
- Find
  \[
  P_0(t, x, z) = P_H(t, x, z)
  \]
  with effective correlation $\rho \to \rho_{xz} \langle f \rangle$
More Formulas!

\[ P_1(t, x, z) = \frac{e^{-r\tau}}{2\pi} \int e^{-ikq} \left( \kappa \theta \hat{f}_0(\tau, k) + z \hat{f}_1(\tau, k) \right) \]
\[ \times \hat{G}(\tau, k, z) \hat{h}(k) dk, \]
\[ \hat{f}_0(\tau, k) = \int_0^\tau \hat{f}_1(t, k) dt, \]
\[ \hat{f}_1(\tau, k) = \int_0^\tau b(s, k) e^{A(\tau, k, s)} ds. \]

\[ b(\tau, k) = - \left( V_1 D(\tau, k) \left( -k^2 + ik \right) + V_2 D^2(\tau, k) (-ik) \right. \]
\[ + V_3 \left( ik^3 + k^2 \right) + V_4 D(\tau, k) \left( -k^2 \right). \]

- \( A(\tau, k, s) \) solves ODE in \( \tau \)
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Heston 2.0
\( P_1 \) is linear function of four constants

\[ V_1 = \rho_{yz} \sigma \nu \sqrt{2} \langle \phi' \rangle, \]
\[ V_2 = \rho_{xz} \rho_{yz} \sigma^2 \nu \sqrt{2} \langle \psi' \rangle, \]
\[ V_3 = \rho_{xy} \nu \sqrt{2} \langle f \phi' \rangle, \]
\[ V_4 = \rho_{xy} \rho_{xz} \sigma \nu \sqrt{2} \langle f \psi' \rangle. \]

\( \psi(y) \) and \( \phi(y) \) solve Poisson equations

Each \( V_i \) has unique effect on implied volatility.
Effect of $V_1$ and $V_2$ on Implied Volatility

- $V_i = 0$ corresponds to Heston
Effect of $V_3$ and $V_4$ on Implied Volatility

- $V_i = 0$ corresponds to Heston

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Captures More Features of Smile

- Better fit for far ITM and OTM European options
- 583 days to maturity
Simultaneous Fit Across Expirations Is Improved

- Vast improvement for short expirations
- 65 days to maturity
Summary

- **Heston model** provides easy way to calculate option prices in stochastic volatility setting, but fails to capture some features of implied volatility surface

- **Multiscale model** offers improved fit to implied volatility surface while maintaining convenience of option pricing formulas
For Further Reading I

Sassan Alizadeh, Michael W. Brandt, and Francis X. Diebold. 
Range-Based Estimation of Stochastic Volatility Models. 

Torben G. Andersen and Tim Bollerslev. 
Intraday periodicity and volatility persistence in financial markets. 

Mikhail Chernov, A. Ronald Gallant, Eric Ghysels, and George Tauchen. 
Alternative models for stock price dynamics. 
For Further Reading II


Jim Gatheral.  
_The Volatility Surface: a Practitioner’s Guide._  

Eric Hillebrand.  
Overlaying time scales and persistence estimation in garch(1,1) models.  
_Econometrics 0301003, EconWPA, January 2003._

Blake D. Lebaron.  
_SSRN eLibrary, 2001._
For Further Reading IV

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  Pricing foreign currency options with stochastic volatility. 

- Ulrich A. Muller, Michel M. Dacorogna, Rakhal D. Dave, 
  Richard B. Olsen, Olivier V. Pictet, and Jacob E. von 
  Weizsacker. 
  Volatilities of different time resolutions – analyzing the 
  dynamics of market components. 
J.E. Zhang and Jinghong Shu.
Pricing standard & poor’s 500 index options with heston’s model.