A Fast Mean-Reverting Correction to Heston's Stochastic Volatility Model

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Outline

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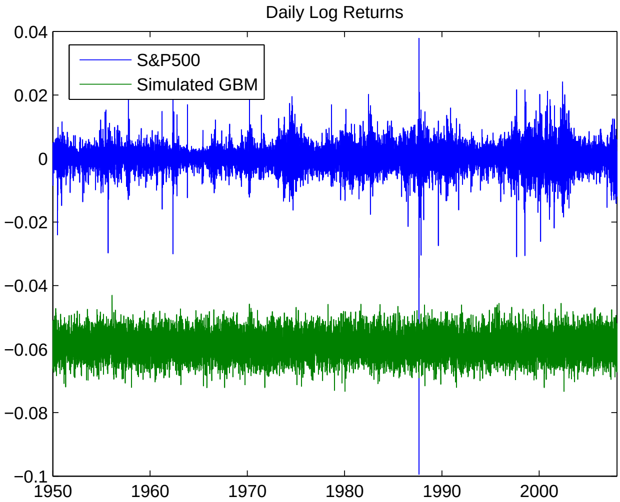
Volatility Not Constant



Heston Under Risk-Neutral Measure

- Motivated by notion that volatility not constant

$$
\begin{aligned}
dX_t &= rX_t dt + \sqrt{Z_t} X_t dW_t^x \\
dZ_t &= \kappa(\theta - Z_t)dt + \sigma \sqrt{Z_t}\, dW_t^z \\
d\left\langle W^x, W^z \right\rangle_t &= \rho dt
\end{aligned}
$$

- One-factor stochastic volatility model
- Square of volatility, Z_t, follows CIR process

Outline

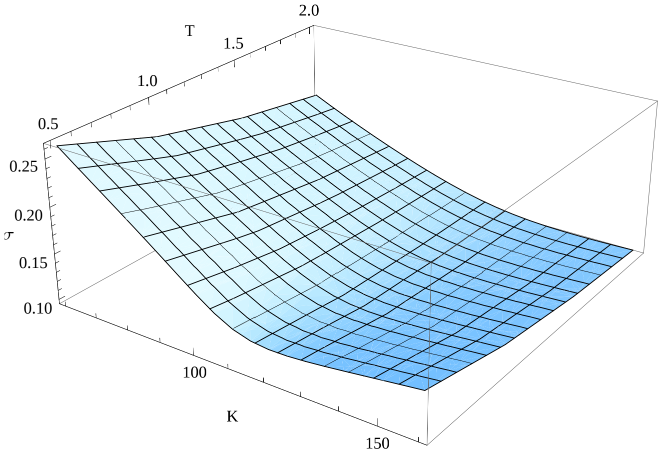
Formulas!

- Explicit formulas for European options:

$$
\begin{aligned}
P_H(t,x,z) &= e^{-r\tau}\frac{1}{2\pi}\int e^{-ikq}\widehat{G}(\tau,k,z)\widehat{h}(k)dk \\
q(t,x) &= r(T-t)+\log x, \\
\widehat{h}(k) &= \int e^{ikq}h(e^q)dq, \\
\widehat{G}(\tau,k,z) &= e^{C(\tau,k)+zD(\tau,k)} \\
&\cdots
\end{aligned}
$$

- $C(\tau,k)$ and $D(\tau,k)$ solve ODE's in $\tau = T - t$.
- Note: audience tunes out if you put too many equations on a slide

Pretty Pictures!



Pretty Pictures Explained

- Implied volatility $\sigma_{Imp}(T, K)$ defined by

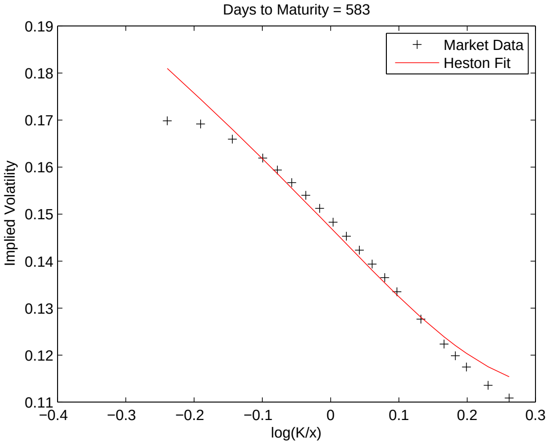
$$P_{BS}(\sigma_{Imp}(T, K)) = P(T, K)$$

  P is price of option with strike K and expiration T
- Heston captures well-documented features of implied volatility surface: smile and skew

Outline

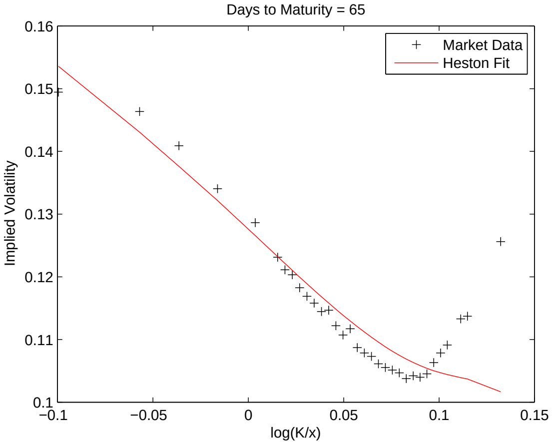
Captures Some . . . Not All Features of Smile

- Misprices far ITM and OTM European options [5] [12]



Simultaneous Fit Across Expirations Is Poor

- Particular difficulty fitting short expirations [7]



Outline

What's Wrong with Heston?

- Single factor of volatility running on single time scale not sufficient to describe dynamics of the volatility process.
- Not just Heston . . . Any one-factor stochastic volatility model has trouble fitting implied volatility levels across all strikes and maturities [7]
- Empirical evidence suggests existence of several stochastic volatility factors running on different time scales

Evidence

- [1] [2] [3] [4] [6] [8] [9] [10] [11]

Outline

Multiscale Under Risk-Neutral Measure

$$
\begin{aligned}
dX_t &= rX_t dt + \sqrt{Z_t}\, f(Y_t) X_t dW_t^x \\
dY_t &= \frac{Z_t}{\epsilon}(m - Y_t)dt + \nu\sqrt{2}\sqrt{\frac{Z_t}{\epsilon}}dW_t^y \\
dZ_t &= \kappa(\theta - Z_t)dt + \sigma\sqrt{Z_t}\, dW_t^z \\
d\left\langle W^i, W^j \right\rangle_t &= \rho_{ij} dt \qquad i,j \in \{x,y,z\}
\end{aligned}
$$

- Volatility controlled by product $\sqrt{Z_t}\, f(Y_t)$
- Y_t modeled as OU process running on time-scale ϵ / Z_t
- Note: $f(y) = 1 \Rightarrow$ model reduces to Heston

Outline

1. Heston Model
   - Motivation and Dynamics
   - Why We like Heston
   - Problems with Heston
2. Multiscale Model
   - Motivation for Multiple Time Scales
   - Multiscale Dynamics
   - Option Pricing
3. Numerical Work
   - Multiscale Implied Volatility Surface
   - Multiscale Fit to Data

Option Pricing PDE

- Price of European Option Expressed as

$$P_t = \mathbb{E}\left[e^{-r(T-t)} h(X_T) \middle| X_t, Y_t, Z_t \right] =: P^\epsilon(t, X_t, Y_t, Z_t)$$

- Using Feynman-Kac, derive following PDE for P^ϵ

$$\mathcal{L}^\epsilon P^\epsilon(t,x,y,z) = 0,$$
$$\mathcal{L}^\epsilon = \frac{\partial}{\partial t} + \mathcal{L}_{(X,Y,Z)} - r\,,$$
$$P^\epsilon(T,x,y,z) = h(x)$$

Some Book-Keeping of $\mathcal{L}^\epsilon$

- $\mathcal{L}^\epsilon$ has convenient form

$$\mathcal{L}^\epsilon = \frac{z}{\epsilon}\mathcal{L}_0 + \frac{z}{\sqrt{\epsilon}}\mathcal{L}_1 + \mathcal{L}_2,$$

$$\mathcal{L}_0 = \nu^2 \frac{\partial^2}{\partial y^2} + (m - y)\frac{\partial}{\partial y}$$

$$\mathcal{L}_1 = \rho_{yz}\sigma\nu\sqrt{2}\,\frac{\partial^2}{\partial y \partial z} + \rho_{xy}\nu\sqrt{2}\,f(y)x\,\frac{\partial^2}{\partial x \partial y}$$

$$\mathcal{L}_2 = \frac{\partial}{\partial t} + \frac{1}{2}f^2(y)zx^2\frac{\partial^2}{\partial x^2} + r\left(x\frac{\partial}{\partial x} - \cdot\right)$$
$$+ \frac{1}{2}\sigma^2 z\frac{\partial^2}{\partial z^2} + \kappa(\theta - z)\frac{\partial}{\partial z} + \rho_{xz}\sigma f(y)zx\frac{\partial^2}{\partial x \partial z}.$$

Perturbative Solution

- PDE has no analytic solution for general $f(y)$
- Perform singular perturbation with respect to ϵ

$$P^{\epsilon} = P_0 + \sqrt{\epsilon} P_1 + \epsilon P_2 + \dots$$

- Turns out P_0 and P_1 functions of t, x, and z only
- Find

$$\boxed{P_0(t,x,z) = P_H(t,x,z)}$$

with effective correlation $\rho \to \rho_{xz} \langle f \rangle$

More Formulas!

$$P_1(t,x,z) = \frac{e^{-r\tau}}{2\pi} \int e^{-ikq} \left(\kappa\theta \widehat{f}_0(\tau,k) + z\widehat{f}_1(\tau,k) \right) \times \widehat{G}(\tau,k,z)\widehat{h}(k)dk,$$

$$\widehat{f}_0(\tau,k) = \int_0^\tau \widehat{f}_1(t,k)dt,$$

$$\widehat{f}_1(\tau,k) = \int_0^\tau b(s,k)e^{A(\tau,k,s)}ds.$$

$$b(\tau,k) = -\left(V_1 D(\tau,k)\left(-k^2 + ik\right) + V_2 D^2(\tau,k)(-ik) + V_3\left(ik^3 + k^2\right) + V_4 D(\tau,k)\left(-k^2\right)\right).$$

- $A(\tau,k,s)$ solves ODE in τ

Outline

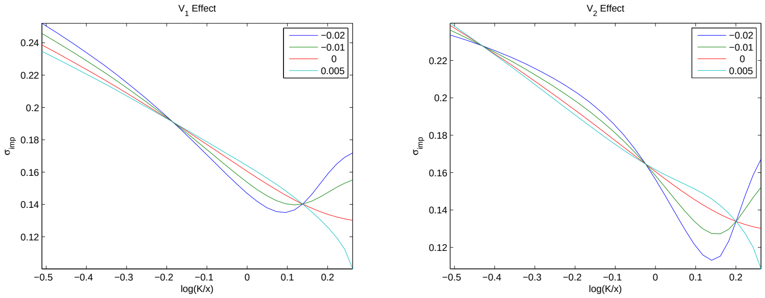
Group Parameters

- P_1 is linear function of four constants

$$
\begin{aligned}
V_1 &= \rho_{yz}\sigma\nu\sqrt{2}\left\langle \phi' \right\rangle , \\
V_2 &= \rho_{xz}\rho_{yz}\sigma^2\nu\sqrt{2}\left\langle \psi' \right\rangle , \\
V_3 &= \rho_{xy}\nu\sqrt{2}\left\langle f\phi' \right\rangle , \\
V_4 &= \rho_{xy}\rho_{xz}\sigma\nu\sqrt{2}\left\langle f\psi' \right\rangle .
\end{aligned}
$$

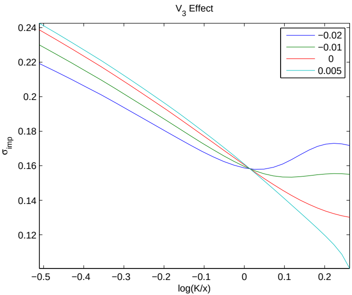
- $\psi(y)$ and $\phi(y)$ solve Poisson equations
- Each V_i has unique effect on implied volatility

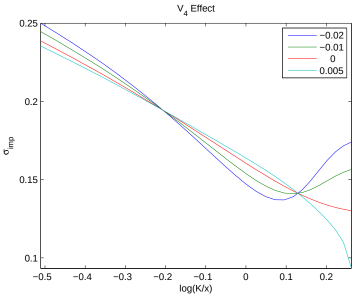
Effect of V_1 and V_2 on Implied Volatility



- $V_i = 0$ corresponds to Heston

Effect of V_3 and V_4 on Implied Volatility



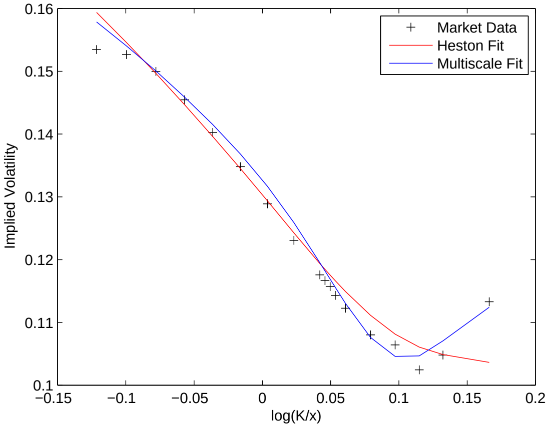


- $V_i = 0$ corresponds to Heston

Outline

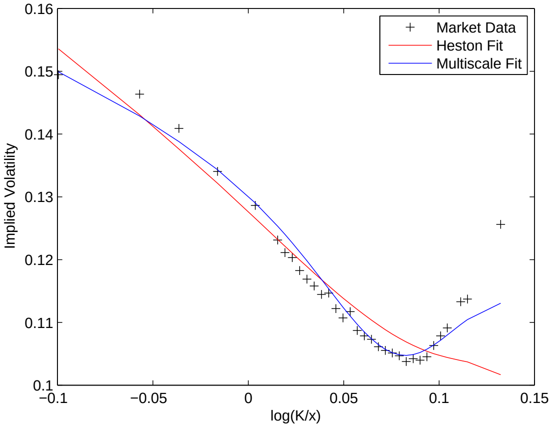
Captures More Features of Smile

- Better fit for far ITM and OTM European options
- 583 days to maturity



Simultaneous Fit Across Expirations Is Improved

- Vast improvement for short expirations
- 65 days to maturity



Summary

- Heston model provides easy way to calculate option prices in stochastic volatility setting, but fails to capture some features of implied volatility surface
- Multiscale model offers improved fit to implied volatility surface while maintaining convenience of option pricing formulas

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