Modeling Default Correlation and Clustering: A Time Change Approach

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  - Nov (1 week span): Landsbanki (4th), Glitnir (5th), Kaupthing Bank (6th)

Default Correlation and Clustering via Multivariate Lévy Subordination

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Consider n obligors with each default time, τ_i, defined by

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$$\Lambda_t^i := \sum_{a=1}^d A_{i,a} \, Z_t^a, \;\; i = 1, ..., n,$$

A is an $n \times d$ matrix with non-negative entries $A_{i,a} \geq 0$

d-dimensional subordinator

d-dimensional subordinator, $\mathcal{T}_t$

Lévy process in $\mathbb{R}_+^d = [0, \infty)^d$ that is non-decreasing in each of its coordinates. That is, each of its coordinates is a one-dimensional Lévy subordinator.

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The (d-dimensional) Laplace transform of a d-dimensional subordinator is given by (here $u_a \geq 0$ and $\langle u, v \rangle = \sum_{a=1}^{d} u_a v_a$)

$$\mathbb{E}[e^{-\langle u, \mathcal{T}_t \rangle}] = e^{-t\phi(u)}$$

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The Laplace exponent $\phi(u)$ is given by the Lévy-Khintchine formula:

$$\phi(u) = \langle \gamma, u \rangle + \int_{\mathbb{R}_+^d} (1 - e^{-\langle u, s \rangle}) \nu(ds),$$

where (drift) $\gamma \in \mathbb{R}_+^d$ and the Lévy measure ν is a sigma-finite measure on $\mathbb{R}^d$ concentrated on $\mathbb{R}_+^d \backslash \{0\}$ such that $\int_{\mathbb{R}_+^d} (|s| \wedge 1) \nu(ds) < \infty$.

d-dimensional subordinator (special case)

Linear Tranformations of Independent Subordinators

Let $\mathcal{S}_t^p$ be$N$ independent one-dimensional subordinators and B a $d \times N$ matrix with non-negative entries $B_{a,p}$. Define

$$\mathcal{T}_t^a = \sum_{p=1}^{N} B_{a,p} \mathcal{S}_t^p, \quad a = 1, ..., d.$$

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Then the $\mathbb{R}_+^d$-valued process $\mathcal{T}_t$ is a d-dimensional subordinator with Laplace exponent given by:

$$\phi(u) = \sum_{p=1}^N \phi_p(v_p), \;\; v_p = \sum_{a=1}^d B_{a,p} u_a$$

where $\phi_p(v)$ are Laplace exponents of N independent one-dimensional subordinators $\mathcal{S}^p$, $p = 1, ..., N$.

Two Firms Case : *no time changes*

Consider the following setup for analyzing two firms:

- Two CIR processes $X_t^i,\ i \in \{1,2\}$:

$$dX_t = \kappa(\theta - X_t)dt + \sigma\sqrt{X_t}dB_t, \quad X_0 = x$$

	$x0$	θ	κ	σ
X_t^1	0.005	0.08	0.13	0.07
X_t^2	0.035	0.013	0.21	0.055

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$$\Lambda_t^2 := 0.65 \ \int_0^t X_u^1 du + 0.35 \ \int_0^t X_u^2 du$$

($Z_t = Y(t) = \int_0^t X_u du$, since we are not using time changes yet)

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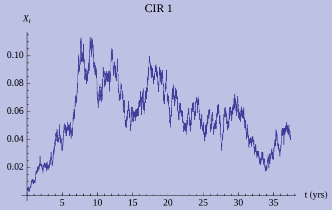
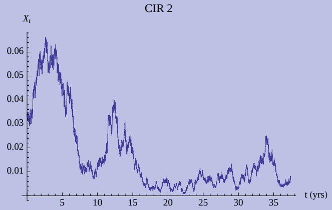
CIR processes





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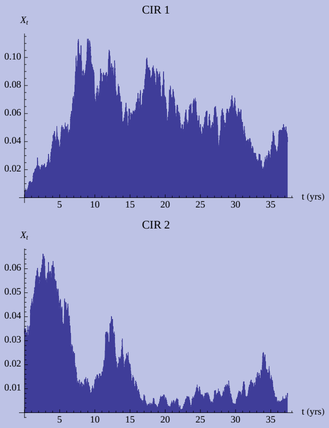

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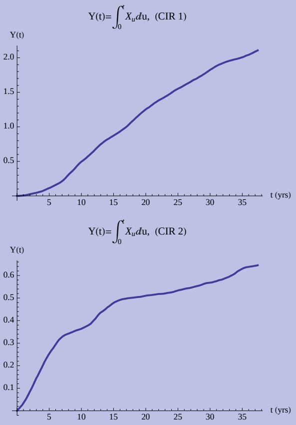

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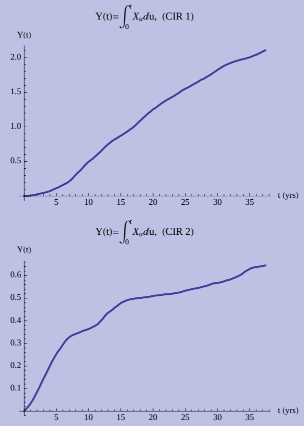

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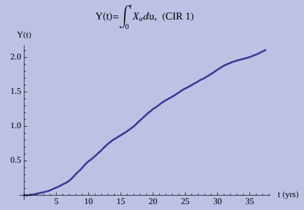

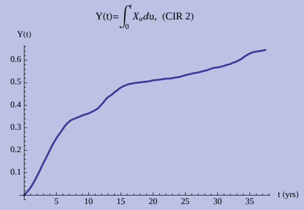

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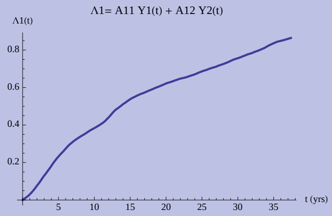

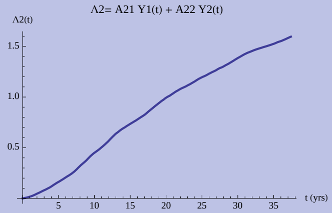

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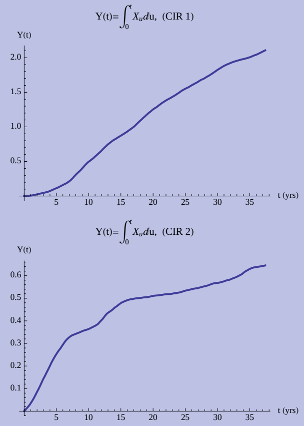

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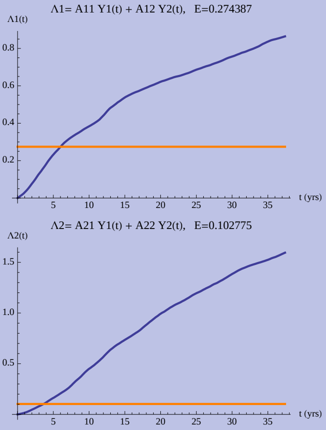

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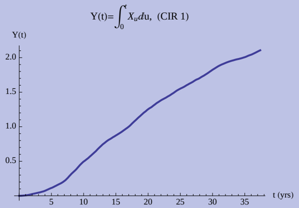

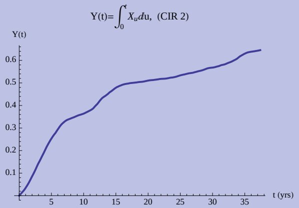

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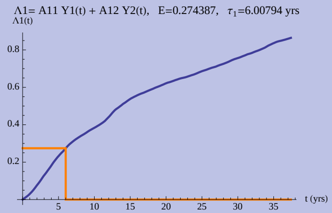

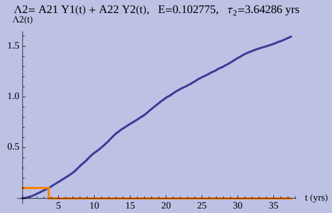

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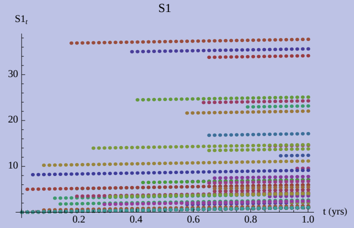

γ	Y	η	C
1	-1	$\frac{1}{10}$	0.005

$E[S_{1yr}]$	$\sigma[S_{1yr}]$	$\alpha = \frac{C}{\eta}$ (arrival)	$\frac{1}{\eta}$ (E.j.size)
1.5	3.16	0.05	10

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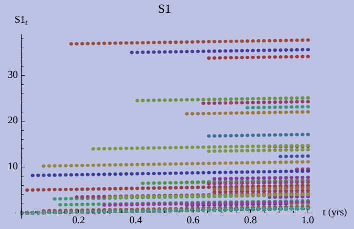

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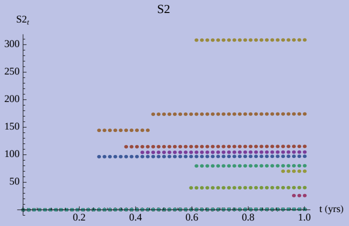
Infrequent large jumps

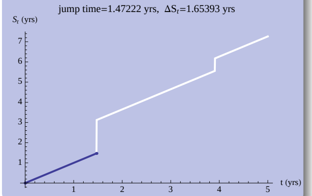


γ	Y	η	C
1	-1	$\frac{1}{100}$	0.00005

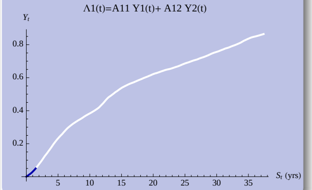
$E[S_{1\text{yr}}]$	$\sigma[S_{1\text{yr}}]$	$\alpha = \frac{C}{\eta}$ (arrival)	$\frac{1}{\eta}$ (E.j.size)
1.5	10.	0.005	100

Two Firms Case: *Subordination*

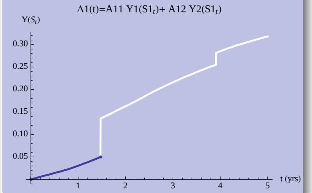
$\mathcal{S}^1_t$ (freq. small jumps)



$A_{11}\ Y^1_t + A_{12}\ Y^2_t$



$\Lambda^1_t = A_{11}\ Y^1_{\mathcal{S}^1_t} + A_{12}\ Y^2_{\mathcal{S}^1_t}$



Recall:

$\Lambda_t = A\, Z_t$ with $Z^i_t = Y^i_{\mathcal{T}^i_t}$

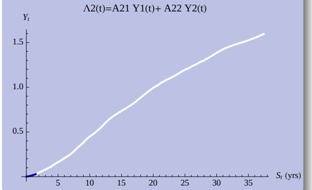
and $\mathcal{T}_t = B\, \mathcal{S}_t$

In this particular case:

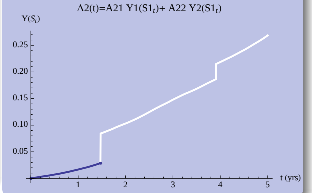
$$A = \begin{pmatrix} 0.15 & 0.85 \\ 0.65 & 0.35 \end{pmatrix},$$

$$B = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$$

$A_{21}\ Y^1_t + A_{22}\ Y^2_t$

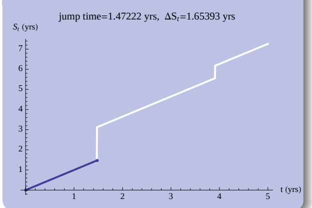


$\Lambda^2_t = A_{21}\ Y^1_{\mathcal{S}^1_t} + A_{22}\ Y^2_{\mathcal{S}^1_t}$

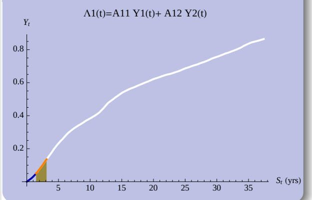


Two Firms Case: *Subordination*

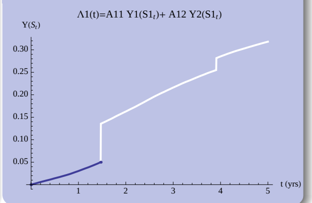
$\mathcal{S}_t^1$ (freq. small jumps)



$A_{11}\, Y_t^1 + A_{12}\, Y_t^2$



$\Lambda_t^1 = A_{11}\, Y^1_{\mathcal{S}_t^1} + A_{12}\, Y^2_{\mathcal{S}_t^1}$



Recall:

$\Lambda_t = A\, Z_t$ with $Z_t^i = Y^i_{\mathcal{T}_t^i}$

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In this particular case:

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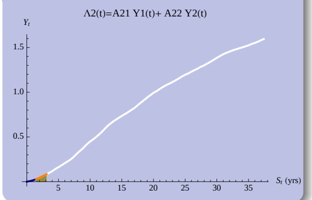
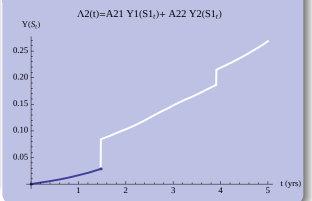


$\Lambda_t^2 = A_{21}\, Y^1_{\mathcal{S}_t^1} + A_{22}\, Y^2_{\mathcal{S}_t^1}$



Two Firms Case: *Subordination*

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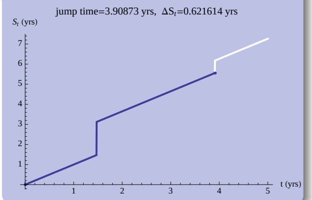

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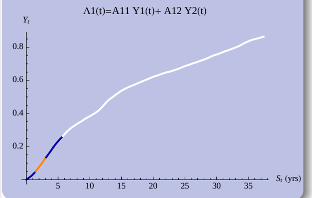

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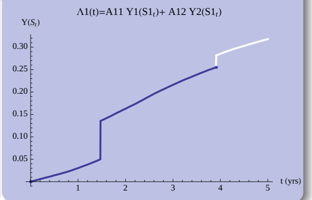

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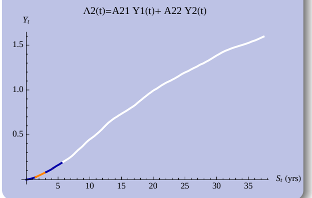

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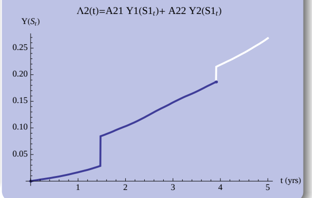
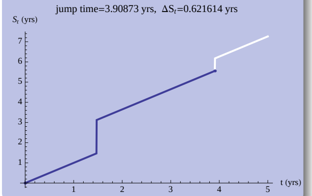


Two Firms Case: *Subordination*

$\mathcal{S}^1_t$ (freq. small jumps)



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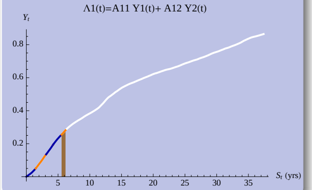

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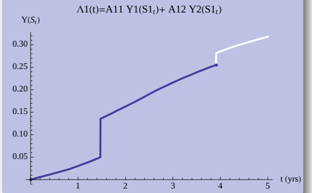

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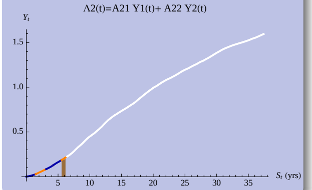

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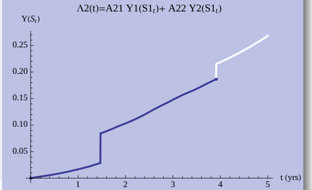
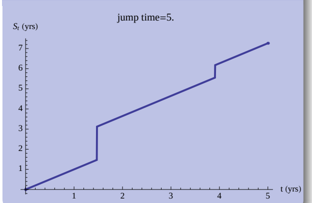


Two Firms Case: *Subordination*

$\mathcal{S}_t^1$ (freq. small jumps)



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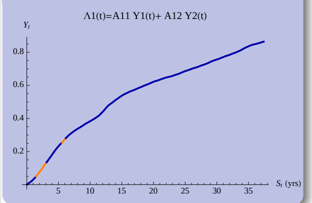
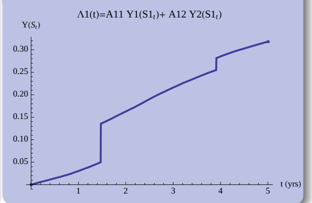


$\Lambda_t^1 = A_{11}\ Y^1_{\mathcal{S}_t^1} + A_{12}\ Y^2_{\mathcal{S}_t^1}$



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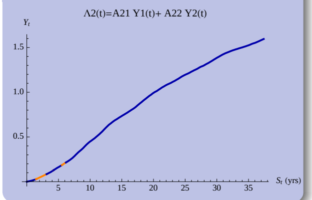
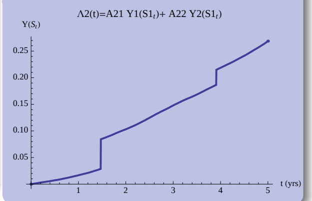


$\Lambda_t^2 = A_{21}\ Y^1_{\mathcal{S}_t^1} + A_{22}\ Y^2_{\mathcal{S}_t^1}$



Two Firms Case: *Subordination*

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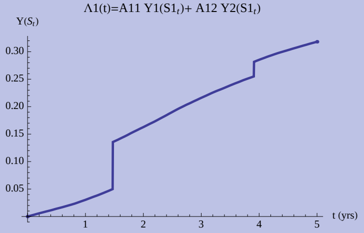

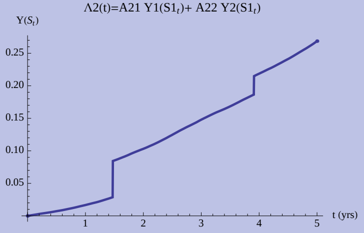

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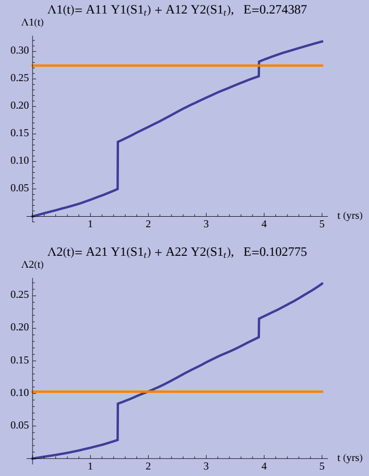
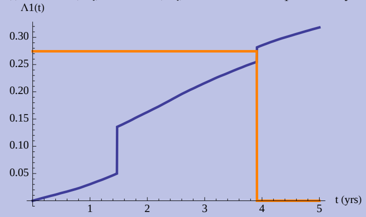


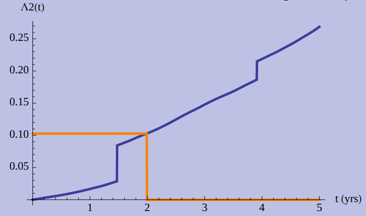
Two Firms Case: *Subordination*

$\mathcal{S}^1_t$ (freq. small), $B = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$

$\Lambda 1(t)$= A11 Y1($S1_t$) + A12 Y2($S1_t$), E=0.274387, τ_1=3.90873 yrs



$\Lambda 2(t)$= A21 Y1($S1_t$) + A22 Y2($S1_t$), E=0.102775, τ_2=1.99206 yrs



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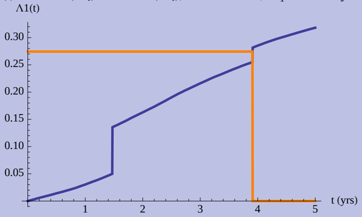

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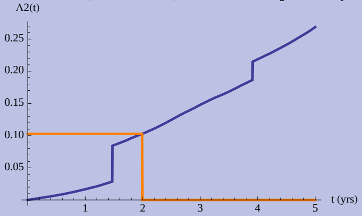

$\mathcal{S}^2_t$ (infreq. large), $B = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$

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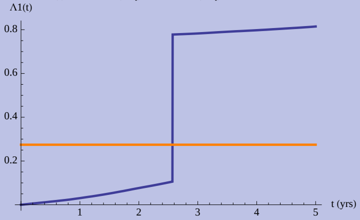

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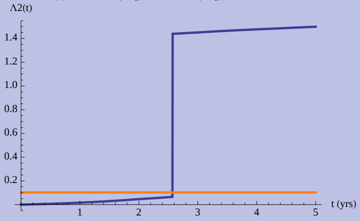

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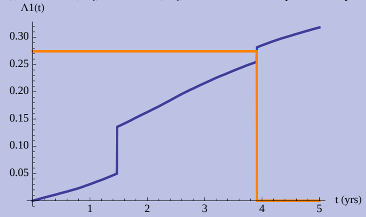

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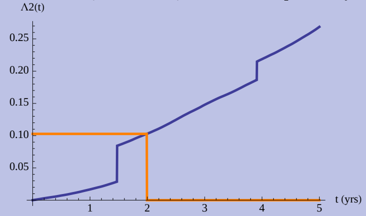
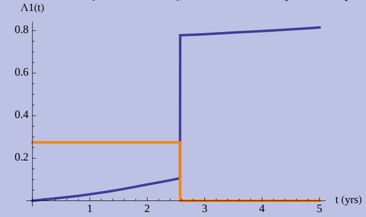


$\mathcal{S}^2_t$ (infreq. large), $B = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$

$\Lambda 1(t)$= A11 Y1($S2_t$) + A12 Y2($S2_t$), E=0.274387, τ_1=2.57143 yrs



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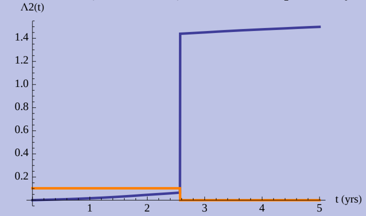

Two Firms Case: *Full Model*

The Default Hazard process is given by,

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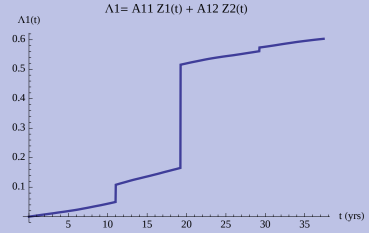

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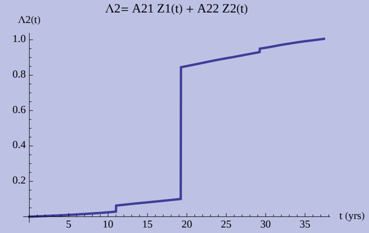

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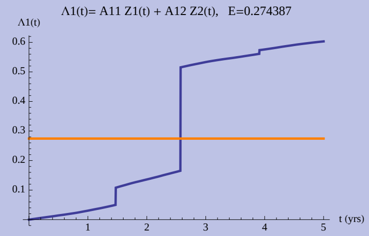

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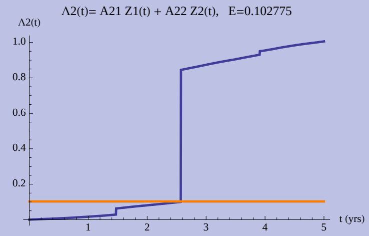

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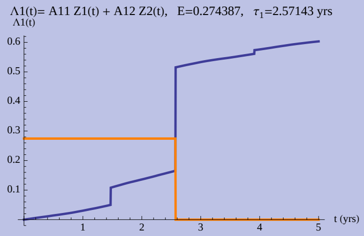

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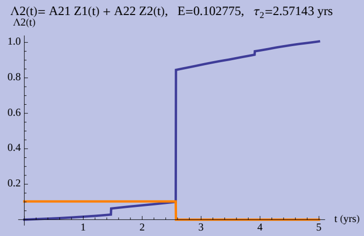

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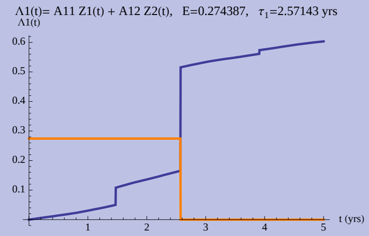

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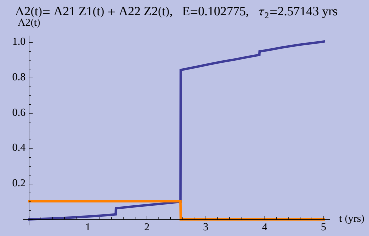

In our framework dependency enters through the *diffusion component, A;* and through the *jump component, B.*

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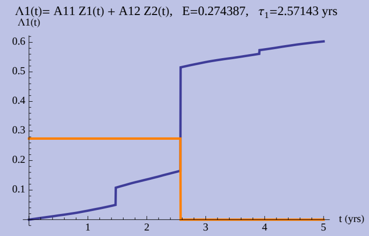

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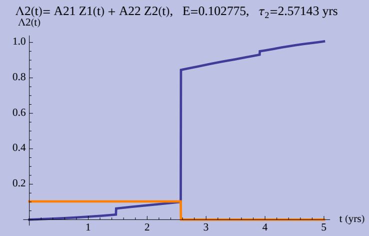

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Non-Trivial Dependency!

Joint Survival Probability

Theorem 1

For $\beta_a \geq 0$, let $\mathcal{L}^a_{x^a,\beta_a}(t)$ denote the Laplace transforms of the integrals up to time t of the Markov processes X^a starting at x^a at time zero:

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For an ordered subset $\Xi = \{i_1, ..., i_k\}$ of $\{1, 2, ..., n\}$ with $1 \leq k \leq n$ define $I_i(\Xi) \in \{0, 1\}$, $i = 1, ..., n$, by:

$$I_i(\Xi) = 1_{\{\Xi\}}(i),$$

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For $\beta_a \geq 0$, let $\mathcal{L}^a_{x^a,\beta_a}(t)$ denote the Laplace transforms of the integrals up to time t of the Markov processes X^a starting at x^a at time zero:

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- Remarkably, the Spectral Method allows us to kill two birds with one stone:
  - Under some additional conditions on Markov processes X, we avoid BOTH the need for the numerical integration, and we only need the Laplace exponent of the subordinator.

Spectral Representation of FK Semigroups of One-Dimensional Diffusions

Feynman-Kac semigroup of linear operators $\{\mathcal{P}_t, t \geq 0\}$

Suppose X_t is a one-dimensional diffusion and consider its F-K semigroup:

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Linear operators $\mathcal{P}_t$ are symmetric in this Hilbert space, that is, $(\mathcal{P}_t f, g) = (f, \mathcal{P}_t g)\ \forall f, g \in L^2((e_1, e_2), \mathfrak{m})$ wrt the *speed measure:*

$$\mathfrak{m}(dx) = \frac{2}{\sigma^2(x)} \exp\left(\int_{x_0}^x \frac{2\mu(y)}{\sigma^2(y)} dy\right) dx$$

where $k(x)$ is the killing rate; whereas, $\mu(x)$ and $\sigma(x)$ are the (state dependent) drift and volatility of the diffusion process X_t, respectively.

Spectral Representation of FK Semigroups of One-Dimensional Diffusions

Spectral Representation

Under some conditions on the behavior of $\mu(x)$, $\sigma(x)$, and $k(x)$ near the boundaries e_1 and e_2, the spectrum is purely discrete and the spectral expansion reduces to the eigenfunction expansion:

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$$\mathcal{G}\varphi_n(x) = \frac{1}{2}\sigma^2(x)\varphi_n''(x) + \mu(x)\varphi_n'(x) - k(x)\varphi_n(x) = -\lambda_n \varphi_n$$

The Feynman-Kac semigroup of the CIR Process.

Let X_t be a CIR diffusion starting from $X_0 = x > 0$ and solving the SDE

$$dX_t = \kappa\left(\theta - X_t\right)dt + \sigma\sqrt{X_t}dB_t,$$

and with the speed measure, $\mathfrak{m}(x) = \frac{2}{\sigma^2}x^{b-1}e^{-\frac{2\kappa}{\sigma^2}x}$.

The spectrum is discrete, and the eigenfunction expansion of the Laplace transform reads ($k(x) = \beta x$ killing rate):,

$$\mathcal{L}_{x,\beta}(t) = \mathbb{E}_x\left[e^{-\beta\int_0^t X_s ds}\right] = \sum_{n=0}^{\infty} c_n e^{-\lambda_n t}\varphi_n(x), \quad c_n = (1,\varphi_n).$$

where the eigenfunctions, eigenvalues and expansion coefficients are given by,

$$\lambda_n = \zeta n + \frac{b}{2}(\zeta - \kappa), \quad \varphi_n(x) = \mathcal{N}_n \exp\left(\frac{\kappa-\zeta}{\sigma^2}x\right)L_n^{(b-1)}\left(\frac{2\zeta}{\sigma^2}x\right),$$

$$c_n = (1,\varphi_n) = \frac{1}{\mathcal{N}_n}\left(\frac{\beta\sigma^2}{\kappa+\zeta}\right)^b\left(\frac{\kappa-\zeta}{\kappa+\zeta}\right)^n,$$

$$\zeta := \sqrt{\kappa^2 + 2\beta\sigma^2}, \quad b := \frac{2\kappa\theta}{\sigma^2}, \quad \mathcal{N}_n = \sqrt{\frac{\beta\sigma^2(n!)}{2\Gamma(b+n)}\left(\frac{2\zeta}{\beta\sigma^2}\right)^{\frac{b}{2}}},$$

$L_n^{(b-1)}$ are the generalized Laguerre polynomials

Spectral Expansion of the Joint Survival Probability

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where $\lambda_n^{a,\Xi}$ and $\varphi_n^{a,\Xi}$ are the eigenvalues and eigenfunctions of the (negative of) the infinitesimal generator of the Feynman-Kac semigroup for the process X^a with $k(x) = \beta_a^{\Xi} x$

x^a is the initial value of the process $X_0^a = x^a$, $c_n^{a,\Xi} = (1, \varphi_n^{a,\Xi})$, and $\phi(u)$ is the Laplace exponent of the subordinator $\mathcal{T}$.

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With the explicit expressions for joint survival probabilities (spectral expansions), we can explicitly compute a variety of dependence measures among default events and times in this class of models. For instance,

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Correlation matrix for default indicators:

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where $P_i(t)$ are the single-name survival probabilities and $P_{ij}(t)$ are the joint survival probabilities for the pairs of names.

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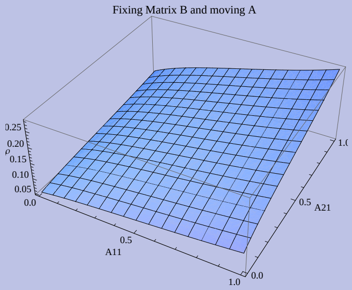
where $\mu_i^\tau = \mathbb{E}[\tau_i]$ and $\sigma_i^\tau = \sqrt{\mathbb{E}[\tau_i^2] - (\mu_i^\tau)^2}$ are the mean and standard deviation of single-name default times.

Correlation of default indicators, $\rho_{ij}^{D}(t)$

Moving the *diffusion component, A*



Moving the Matrix A:

$$A = \begin{pmatrix} A_{11} & 1 - A_{11} \\ A_{21} & 1 - A_{21} \end{pmatrix}$$

Fixing the Matrix B:

$$B = \begin{pmatrix} 0.5 & 0.5 \\ 0.7 & 0.3 \end{pmatrix}$$

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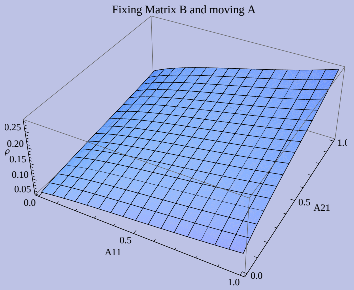

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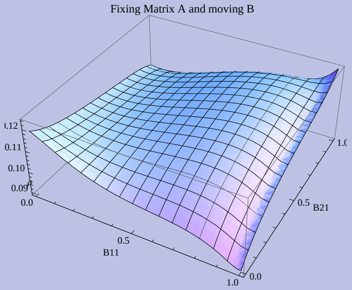

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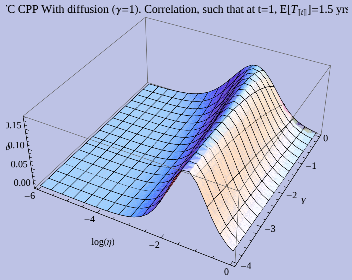
$\mathcal{S}_t^1 = \gamma t + CPP$, $(\gamma = 1)$



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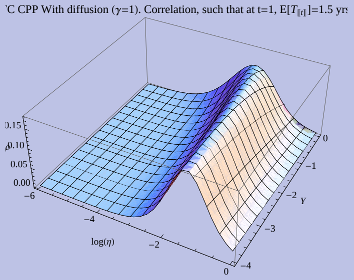

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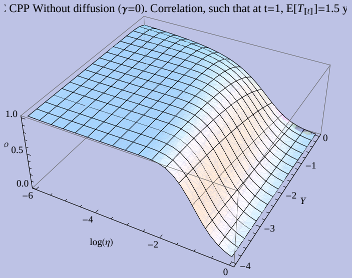

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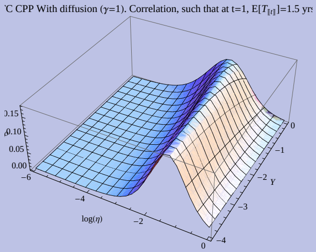

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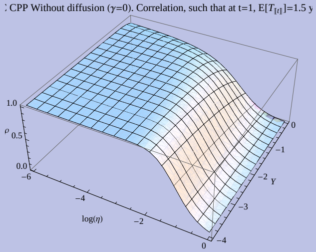

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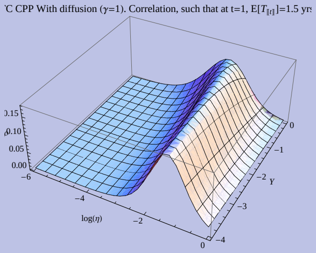

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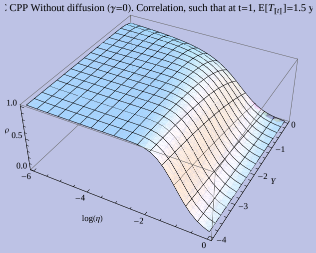

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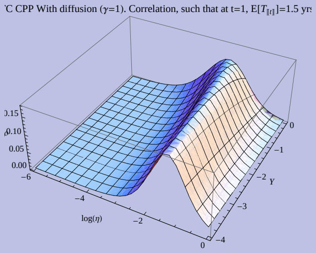

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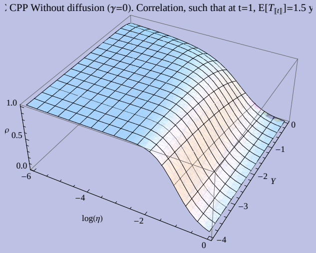

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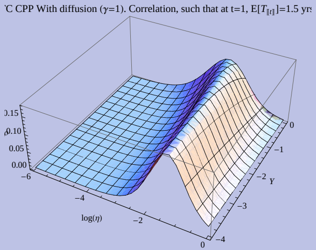

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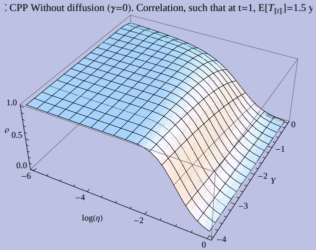

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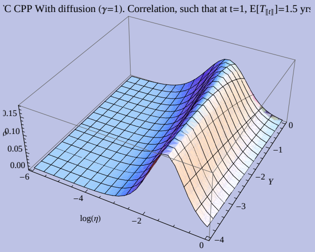

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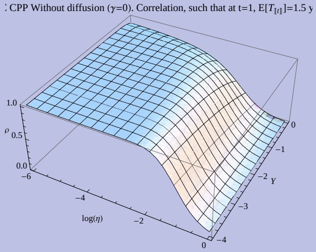

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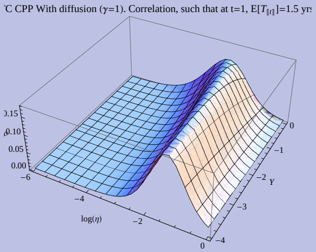

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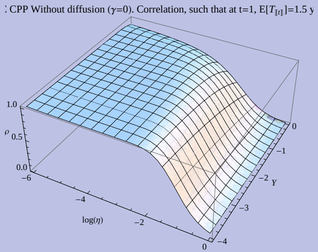

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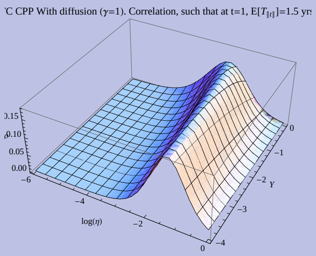

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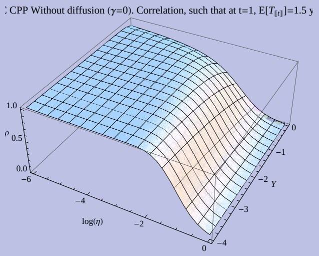

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- **Any correlation level can be achieved by using linear combinations of subordinators with different Lévy specifications**

Cloncluding Remarks

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- This is a work in progress and we are currently generating numerical examples for the pricing of Credit Swap Baskets and CDO's

Questions?

Thank you