Analysis of continuous strict local martingales via h-transforms

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Abstract

We study strict local martingales via h-transforms, a method which first appeared in Delbaen-Schachermayer. We show that strict local martingales arise whenever there is a consistent family of change of measures where the two measures are not equivalent to one another. Several old and new strict local martingales are identified. We treat examples of diffusions with various boundary behavior, size-bias sampling of diffusion paths, and non-colliding diffusions. A multidimensional generalization to conformal strict local martingales is achieved through Kelvin transform. As curious examples of non-standard behavior, we show by various examples that strict local martingales do not behave uniformly when the function $(x-K)^+$ is applied to them. Implications to the recent literature on financial bubbles are discussed.

1 Introduction

Local martingales which are not martingales (known as "strict" local martingales) arise naturally in the Doob-Meyer decomposition and in multiplicative functional decompositions, as well as in stochastic integration theory. They are nevertheless often considered to be anomalies, processes that need to be maneuvered by localization. Hence, studies focussed purely on strict local martingales are rare. One notable exception is the article by Elworthy, Li, & Yor [8] who study their properties in depth. On the other hand, applications of strict local martingales are common. See the articles by Bentata & Yor

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[2], Biane & Yor [3], Cox & Hobson [4], Fernholz & Karatzas [11], and the very recent book-length preprint of Profeta, Roynette, and Yor [28].

Our goal in this paper is to demonstrate that strict local martingales capture a fundamental probabilistic phenomenon. A first example occurs when there is a pair of probability measures where one strictly dominates the other (in the sense that their null sets are not the same). For positive local martingales, such a phenomenon was originally identified by Delbaen and Schachermayer in [6]. We start with the following result.

Let $\left(\Omega, \{\mathcal{F}_t\}_{t\geq 0}\right)$ be a filtered sample space on which two probability measures P and Q are defined. We assume that P is locally strictly dominated by Q, in the sense that P is absolutely continuous with respect to Q ($P << Q$) on every $\mathcal{F}_t$ (see, for example, the book by Jacod & Shiryaev [15] for a systematic treatment of this idea) and if we let $dP/dQ \mid_{\mathcal{F}_t} := h_t$, then we have

$$Q\left(\tau_0 < \infty\right) > 0, \tag{1}$$

where $\tau_0 = \inf\left\{s > 0,\ h_s = 0\right\}$. We claim the following result.

Proposition 1. *Assume that h is a continuous process Q-almost surely. Let $\{f_t,\ t \geq 0\}$ be a continuous Q-martingale adapted to the filtration $\{\mathcal{F}_t\}$. Suppose either of the two conditions hold:*

(i) f is uniformly integrable, $E^Q(f_0) \neq 0$, and $Q\left(\tau_0 < \infty\right) = 1$, or,

(ii) f is nonnegative, $Q(\sigma_0 > \tau_0) > 0$, where $\sigma_0 = \inf\{s \geq 0 : f_s = 0\}$.

Then the following process

$$N_t := \frac{f_t}{h_t}$$

is a strict local martingale under P.

However, if f is nonnegative and $Q(\sigma_0 > \tau_0) = 0$, the process N_t is a true martingale.

We identify several strict local martingales as an application of the previous result, in diverse topics such as diffusions conditioned to exit through a subset of the boundary of a domain, size-biased sampling of diffusion paths, and non-colliding diffusions such as Dyson's Brownian motion from Random Matrix Theory. Each of these examples involve a change of measure that is locally strictly dominated, and hence leads to a plethora of examples of strict local martingales.

A partial converse to Proposition 1 can be easily obtained by extending the argument of [6] and has been done in Proposition 6. We show how stochastic calculus with respect to strict local martingales (which can be quite tricky), can be reduced to stochastic calculus with respect to actual martingales via such a change of measure. Our results in this direction are related to recent work by Madan & Yor [23].

We also prove a multidimensional analogue of our results where a strict local martingale in one-dimension is replaced by a conformal local martingale in three or more dimensions where at least one coordinate process is strict. The analysis exploits the Kelvin transform from classical potential theory.

A convex function applied to a martingale always gives rise to a submartingale. However, this is not always true for strict local martingales. From the standpoint of applications we are interested in two specific functions $x \mapsto (k-x)^+$ and $x \mapsto (x-k)^+$, for some positive k. Both are convex but display very different behavior when applied to strict local martingales. The former always leads to a submartingale, while the latter, although a local submartingale, can have both increasing and decreasing expectations. We analyze its effect on the inverse 3-dimensional Bessel process (the canonical continuous strict local martingale, much as Brownian motion is the canonical continuous martingale) and demonstrate a curious phase transition phenomenon. Similar examples were also identified in [8].

The final part of the paper discusses the implication of our results in mathematical finance. A natural source for local martingales in mathematical finance is the condition of No Free Lunch with Vanishing Risk (see Dealbaen & Schachermayer [5]). Roughly, it states that in a financial market the no arbitrage condition is equivalent (in the case of continuous paths) to the existence of an (equivalent) "risk neutral" probability measure Q which turns the price process into either a martingale or a strict local martingale. The implications of our results can be readily understood if we assume that the risk neutral measure produces a (one-dimensional) price process $X = (X_t)_{t \geq 0}$ that is a strict local martingale. In that case the process $Y_t = (X_t - K)^+$ *need not be a submartingale*, and the function $t \mapsto E\{(X_t - K)^+\}$ need no longer be increasing, contradicting the usual wisdom in the theory. The original purpose of this paper was to understand this phenomenon better, motivated in particular by the role local martingales play in financial bubbles (cf [16] and [17]). We are able to construct an example where Merton's famous mathematical finance "no early exercise" theorem [24] does not hold. Our results indicate that one theoretically possible way to detect a bubble is to analyze the behavior of European call prices through time.

2 A method to generate strict local martingales

We start with the proof of Proposition 1.

Proof of Proposition 1. Let us first show that N is a local martingale. Consider the sequence of stopping times

$$\sigma_k := \inf\{s \geq 0 : \; h_s \leq 1/k\}.$$

Then, it is clear that $P(\lim_{k\to\infty} \sigma_k = \infty) = 1$. Also, by the continuity of h, at the stopping time σ_k, it takes value $1/k$. Thus, for any bounded stopping time τ, we get by the change of measure formula

$$E^P(N_{\tau\wedge\sigma_k}) = E^Q\left(h_{\tau\wedge\sigma_k}\frac{1}{h_{\tau\wedge\sigma_k}}f_{\tau\wedge\sigma_k}\right) = E^Q(f_0).$$

Since $N_{\tau\wedge\sigma_k}$ has the same expectation for all bounded stopping times τ, it follows that $N_{\cdot\wedge\sigma_k}$ is a martingale. The local martingale property now follows.

To show now that it is not a martingale, we compute the expectation of N_t. Note that, since h is a nonnegative Q-martingale, zero is an absorbing state for the process. Thus, again applying the change of measure formula, we get

$$E^P(N_t) = E^Q\left(f_t 1_{\{\tau_0 > t\}}\right), \tag{2}$$

where τ_0 is the hitting time of zero for h.

Now suppose (i) $\{f_t\}$ is a uniformly integrable martingale and $Q(\tau_0 < \infty) = 1$. By uniformly integrability we see from (2) that

$$\lim_{t\to\infty} E^P(N_t) = 0$$

which shows that N is not a martingale, since $E^P(N_0) = E^Q(f_0)$ is assumed to be non-zero.

Finally suppose (ii) $Q(\sigma_0 > \tau_0) > 0$ holds. Since zero is an absorbing state for the nonnegative martingale f, by (1) there is a time $t > 0$ such that $Q(\{f_t > 0\} \cap \{\tau_0 \leq t\}) > 0$. For that particular t, we get from (2) that

$$E^P(N_t) < E^Q(f_t) = E^Q(f_0) = E^P(N_0).$$

This again proves that the expectation of N_t is not a constant. Hence it cannot be a martingale.

The final assertion follows from (2) by noting that N, by virtue of being a nonnegative local martingale, is supermartingale which has a constant expectation. Thus it must be a martingale. $\square$

It is clear that some condition is necessary for the last theorem to hold as can be seen by taking $f_t \equiv h_t$ which results in a true martingale $N_t \equiv 1$.

Several examples of old and new strict local martingales follow from the previous result by suitably choosing a change of measure and the process f. We describe some classes of examples below.

2.1 Diffusions with different boundary behaviors

One of the earliest uses of change of measures was to condition a diffusion to exhibit a particular boundary behavior. This typically involves a change of measure that is locally strictly dominated. We give below two examples to show how local martingales arise from such a set-up.

Example 1. The 3-dimensional Bessel process, BES(3), (see Karatzas & Shreve [19, page 158] for the details) is the (strong) solution of the stochastic differential equation:

$$dX_t = \frac{1}{X_t}dt + d\beta_t, \qquad X_0 = x_0, \tag{3}$$

where β is an one-dimensional standard Brownian motion. It is also the law of the Euclidean norm of a three-dimensional Brownian motion. Since the origin is polar for the three dimensional BM, the reciprocal of this process is well-defined throughout. This reciprocal process, known as the *inverse Bessel process*, serves as a prototypical example of a local martingale which is not a martingale. The strictness holds in spite of the fact that the family of random variables $\{1/X_t,\ t \geq 0\}$ is uniformly integrable on the entire range of t.

This is an immediate example of Proposition 1. To see this let us call as the *canonical space*, the space of continuous functions $C[0,\infty)$ together with the right-continuous filtration obtained from the natural filtration generated by the coordinate process. The laws of all continuous stochastic processes are probability measures on this space.

Let X_t denote the coordinate process on the canonical space, and let Q denote the Wiener measure such that $Q(X_0 = 1) = 1$. Let τ_0 denote the first hitting time of zero, i.e., $\tau_0 = \inf\{t \geq 0,\ X_t = 0\}$. Then $X_{t\wedge\tau_0}$ is a martingale under Q and $E(X_{t\wedge\tau_0}) = 1$. Define a probability measure P by the domination relation

$$\left.\frac{dP}{dQ}\right|_{\mathcal{F}_t} = X_{t\wedge\tau_0}. \tag{4}$$

Then, it follows from Girsanov's theorem ([30, page 327]), that under P, the law of the coordinate process X_t is BES(3) with $X_0 = 1$.

Now that $1/X_t$ is a strict local martingale under P is a corollary of Proposition 1 by taking the process $f_t \equiv 1$ and by noting that f is uniformly integrable and $Q(\tau_0 < \infty) = 1$.

Several other examples of the same spirit can be derived. In particular, for any Bessel process X of dimension $\delta > 2$, it is well-known that $X^{2-\delta}$ is a strict local martingale. This can be proved similarly as above using the martingale $Y^{\delta-2}$, where Y is a Bessel process of dimension $(4-\delta)$ (which can be negative) absorbed at zero. See the article [2] for the details.

Example 2. The previous example can be easily extended to a multidimensional form. Let D be an open bounded domain in $\mathbb{R}^n$ $(n \geq 2)$ where every point on the boundary is regular (in the sense of [19, p. 245]). Consider an n-dimensional Brownian motion X starting from a point $x_0 \in D$ getting absorbed upon hitting the boundary of D, say ∂D. Let Q denote the law of the process $\{X_{t\wedge\tau_D},\ t \geq 0\}$. For any bounded measurable nonnegative function u on ∂D, one can construct the following function

$$f(x_0) = E^Q\left(u\left(X_{\tau_D}\right) \mid X_0 = x_0\right). \tag{5}$$

By the Markov property, it follows that $f(X_{t\wedge\tau_D})$ is a martingale. In fact, f is the solution of the Dirichlet problem on D with boundary data u.

Let B_1 be a connected (nontrivial) proper subset of the boundary ∂D, and let the function u be one on B_1 and zero elsewhere, i.e, $u(x) = 1_{(x\in B_1)}$. In that case, the resulting harmonic function in D is given by

$$v(x_0) = Q\left(X_{\tau_D} \in B_1 \mid X_0 = x_0\right).$$

Again, by the tower property, $h(X_{t\wedge\tau_D}) = v(X_{t\wedge\tau_D})/v(x_0)$ is a nonnegative martingale starting from one. Let P denote the law defined by the domination relation

$$dP/dQ\Big|_{\mathcal{F}_t} = h(X_{t\wedge\tau_D}).$$

This gives us a process law which can be interpreted as *Brownian motion, starting from x_0, conditioned to exit through B_1.*

As a corollary of Proposition 1, we get the following result.

Proposition 2. *Let u be any nonnegative function on ∂D such that $u > 0$ on some subset of $\partial D \backslash B_1$ of positive Lebesgue measure. Let f be the harmonic extension of u given by the formula* (5)*. Then the process*

$$N_t = \frac{f(X_{t\wedge\tau_D})}{v(X_{t\wedge\tau_D})}$$

is a strict local martingale under P.

Proof. Follows from Proposition 1 by verifying condition (ii) since $u(X_{\tau_D})$ need not always be zero when $X_{\tau_D} \notin B_1$. $\square$

2.2 Size-biased sampling of diffusion paths

Another class of interesting examples follow in the case of the size-biased change of measure. Size-biased sampling has been often discussed in connection with discrete distributions, see, for example, the article by Pitman [27] and the references within. It usually involves a finite or countable collection of numbers $\{p_1, p_2, p_3, \ldots\}$ such that each $p_i \geq 0$ and $\sum_i p_i = 1$. A size-biased sample from this collection refers to a sampling procedure where the sample $\tilde{p}$ has the distribution

$$P\left(\tilde{p} = p_i\right) = p_i, \quad \text{for all } i = 1, 2, \ldots.$$

One can now remove this chosen sample from the collection, renormalize it, and repeat the procedure. This is closely connected to urn schemes where each p_i refers to the proportion of balls of a color i that is in an urn. If one randomly selects a ball, the color of the chosen ball has the size-biased distribution.

One can similarly develop a concept of size-biased sampling of diffusion paths as described below. Consider n non-negative diffusions $\{X_1, \ldots, X_n\}$ running in time. Fix a time t, and look at the paths of the diffusions during the time interval $[0,t]$. Denote these random continuous paths by $X_1[0,t], X_2[0,t], \ldots, X_n[0,t]$. Sample one of these random paths with a probability proportional to the terminal value $X_i(t)$. That is, the sampled path has the law

$$Y[0,t] = X_i[0,t], \quad \text{with probability } \frac{X_i(t)}{\sum_{j=1}^n X_j(t)}.$$

How can one describe the law of Y ? In general the law of Y might not be consistently defined as time varies. Nevertheless there are cases where it makes sense. We show below the example when each X_i is a Bessel square process of dimension zero (BESQ^0). This is the strong solution of the SDE

$$Z_t = z + 2\int_0^t \sqrt{Z_s}\, d\beta_s, \qquad z > 0,$$

where β is a one-dimensional Brownian motion. BESQ^0 is a nonnegative martingale also known as the Feller branching diffusion, in the sense that it represents the total surviving population of a critical Galton-Watson branching process. Indeed, our treatment here of size-biased transforms of BESQ processes is inspired by size-biased transforms of Galton-Watson trees (see the work by Lyons-Pemantle-Peres [22] and references to the prior literature referred to there).

The interpretation of such dynamic size-biased sampling when every X_i is a BESQ^0 is straightforward. When each X_i has the law of Feller's branching diffusion, they represent a surviving population from n critical branching processes. We can do size-biased sampling at different time points from these populations. The construction below describes the joint law of these samples.

Consider the canonical sample space for multidimensional diffusions, i.e., the n-dimensional continuous path space $C^n[0,\infty)$, coupled with the usual right-continuous and complete filtration generated by the coordinate maps. Let $\omega_t = (\omega_t(1), \omega_t(2), \ldots, \omega_t(n)$ denote a sample path. Also, as before, we take $\omega[0,t]$ to denote the path during time-interval $[0,t]$. Let Q be the joint law of n independent BESQ^0 processes starting from positive points $(z(1), \ldots, z(n))$. To keep matters simple, we assume all the $z(i)'s$ are the same and equal to z. This induces the following exchangeability property.

Let $T_i : C^n[0,\infty) \to C^n[0,\infty)$ be an operator on the sample space such that $T_i\omega(1) = \omega(i)$ and

$$T_i\omega(j+1) = \omega(j), \quad \text{if } j < i, \quad \text{and,} \quad T_i\omega(j) = \omega(j), \qquad \text{if } j > i. \tag{6}$$

Thus, T_i puts the ith coordinate as the first, and shifts the others appropriately. By our assumed exchangeability under Q, each $T_i\omega$ has also the same law Q. To define the size-biased change of measure we need the following lemma.

Lemma 3. *For any i, the process*

$$M_t(i) = \frac{\omega_t(i)}{\omega_t(1) + \ldots + \omega_t(n)}, \qquad t \geq 0,$$

is a martingale under Q.

Proof. Without loss of generality take $i = 1$. Let $Z(1), Z(2), \ldots, Z(n)$ be independent BESQ^0. The sum $\zeta = Z(1) + \ldots + Z(n)$ is another BESQ^0 process. Thus by Itô's rule we get

$$\begin{aligned} d\left(Z_t(1)/\zeta_t\right) &= Z_t(1) d\left(\frac{1}{\zeta_t}\right) + \frac{1}{\zeta_t} dZ_t(1) + d\left\langle Z_t(1), \zeta_t^{-1}\right\rangle \\ &= \text{local martingale } + \frac{Z_t(1)}{\zeta_t^3} 4\zeta_t dt - \frac{4Z_t(1)}{\zeta_t^2} dt. \end{aligned}$$

This proves that the ratio process is a local martingale. But since it is bounded, it must be a martingale. □

Define the size-biased sampling law P on $(C^n[0,\infty), \{\mathcal{F}_t\})$ by

$$P(A) = E^Q\left[\sum_{i=1}^n \frac{\omega_t(i)}{\omega_t(1)+\cdots+\omega_t(n)} 1_{\{T_i\omega[0,t]\in A\}}\right], \quad \text{for all } A \in \mathcal{F}_t. \quad (7)$$

Note that, by Lemma 3, P defines a consistent probability measure on the filtration $\{\mathcal{F}_t\}$. We are now going to show that P is strictly locally dominated by Q and compute the Radon-Nikodým derivative.

To see that, recall that, under Q, each $T_i\omega$ has the same law Q. Thus, we can simplify expression (7) to write

$$P(A) = nE^Q\left[\frac{\omega_t(1)}{\omega_t(1)+\cdots+\omega_t(n)} 1_{\{\omega[0,t]\in A\}}\right], \quad \text{for all } A \in \mathcal{F}_t.$$

This proves that $P << Q$ and the Radon-Nikodým derivative is given by

$$h_t = \frac{n\omega_t(1)}{\omega_t(1)+\cdots+\omega_t(n)}.$$

Since under the BESQ^0 law, every coordinate can hit zero and get absorbed, the above relation is a locally strict domination and hence leads to examples of strict local martingales.

As an immediate corollary of Proposition 1 we get the following result.

Proposition 4. *Let $(Z(1),\ldots,Z(n))$ be continuous processes whose law is the size-biased sampled BESQ^0 law P described above. Let $\zeta_t = Z_t(1)+\ldots+Z_t(n)$ be the total sum process. Then the processes*

$$N_t = \frac{\zeta_t^2}{Z_t(1)}, \quad U_t = \frac{Z_t(2)\zeta_t}{Z_t(1)}, \quad V_t = \frac{\zeta_t}{Z_t(1)}\prod_{i=2}^n Z_t(i), \quad t \geq 0,$$

are all strict local martingales. However, the process

$$M_t = \zeta_t \prod_{i=2}^n Z_t(i), \quad t \geq 0,$$

is a true martingale.

Proof. Let $Z(1),\ldots,Z(n)$ be iid BESQ^0 processes starting from $z > 0$. Then ζ, $Z(2)$, and $\prod_{i=2}^n Z_t(i)$ are all true martingales which can remain positive when $Z_t(1) = 0$. The result now follows from Proposition 1, condition (ii).

On the other had, the process $\prod_{i=1}^n Z_t(i)$ is a martingale that always hits zero before $Z_t(1)$. Thus, M is a true martingale. □

2.3 Non-colliding diffusions

Our third class of examples are cases when the change of measure leads to non-intersecting paths of several linear diffusions. Probably the most important example of this class is Dyson's Brownian motion which is a solution of the following n-dimensional SDE:

$$d\lambda_t(i) = \sum_{j \neq i} \frac{2}{\lambda_t(i) - \lambda_t(j)} dt + dB_t(i), \quad t \geq 0, \quad i = 1, \ldots, n. \tag{8}$$

Here $(B(1), \ldots, B(n))$ is an n-dimensional Brownian motion. It appears in the context of Random Matrix Theory. Please see the survey article by König [20] and the original paper by Dyson [7] for the details (including the definition of the Gaussian Unitary Ensemble) and the proofs.

Theorem 1. *For any $i = 1, 2, \ldots, n$, and $j > i$, let $\{M_t(i,i),\ t \geq 0\}$, $\{M_t^R(i,j),\ t \geq 0\}$, and $\{M_t^I(i,j),\ t \geq 0\}$ be independent real standard Brownian motions, starting at zero. The Hermitian random matrix $M_t = (M_t(i,j),\ 1 \leq i,j \leq n)$, with*

$$M_t(i,j) = M_t^R(i,j) + iM_t^I(i,j), \quad i < j,$$

has the distribution of the Gaussian Unitary Ensemble at time $t = 1$. Then the process $(\lambda_t,\ t \geq 0)$ of n eigenvalues of M_t satisfies SDE (8). It can interpreted as a conditional Brownian motion in $\mathbb{R}^n$, starting at zero, conditioned to have

$$\lambda_t(1) < \lambda_t(2) < \cdots < \lambda_t(n), \qquad \text{for all} \quad t > 0.$$

What is interesting is that the process in (8) can be obtained as a strict local domination relation from the n-dimensional Wiener measure using the harmonic function

$$\Delta_n(x) = \prod_{1 \leq i < j \leq n} (x(j) - x(i)), \qquad x = (x(1), \ldots, x(n)),$$

which is the well-known Vandermonde determinant. That the function Δ_n is harmonic can be found in [20, p. 433] where it is shown that $\Delta_n(W_t)$ is a martingale when W is an n-dimensional Brownian motion and that the law of the process in (8) (say P) can be obtained from the n-dimensional Wiener measure Q, by using $h_t = \Delta_n(W_t)$ as the Radon-Nikodým derivative. Since $\Delta_n(W)$ is zero whenever any two Brownian coordinates $W(i), W(j)$ are equal ("collide"), we are in the scenario of Proposition 1. We prove the following result.

Proposition 5. *Consider Dyson's Brownian motion $(\lambda_t(1), \ldots, \lambda_t(n))$ in* (8)*, and for some $m < n$ consider the process*

$$\Delta_m(t) = \prod_{1\le i<j\le m} (\lambda_t(j) - \lambda_t(i)), \qquad t \ge 0.$$

Then the process $N_t = \Delta_m(t)/\Delta_n(t)$ is a strict local martingale. As a consequence if we consider the Vandermonde matrix-valued process

$$A_t = (\lambda_t^{j-1}(i),\ i,j = 1,2,\ldots,n), \qquad t \ge 0,$$

Then every process $A_t^{-1}(n,i)$ is a strict local martingale for $i = 1,2,\ldots,n$.

Proof. Note that, if W is an n-dimensional Brownian motion then $\Delta_m(W_t)$, for any $m \le n$, is a true martingale. Since $m < n$, it is possible to have $\Delta_m(W_t)$ to be positive when $\Delta_n(W_t) = 0$. The result that N is a strict local martingale now follows from Proposition 1.

For the second assertion we use adjoint relation for the inverse. Let $\det(B)$ for a square matrix B refer to its determinant. Then

$$A_t^{-1}(n,i) = \frac{(-1)^{i+n}}{\det(A_t)} \det\left(\hat{A}_t(i,n)\right), \quad i = 1,2,\ldots,n,$$

where $\hat{A}_t(i,n)$ is the matrix obtained from A_t by removing the ith row and the nth column.

Now, A_t is the Vandermonde matrix, so its determinant is equal to $\Delta_n(t)$. If we remove the ith row and the nth column from A_t we get an $(n-1)\times(n-1)$ order Vandermonde matrix of all the λ_j's except the ith. Its determinant is again a Vandermonde determinant. Each of the ratios

$$\frac{\det\left(\hat{A}_t(i,n)\right)}{\det(A_t)},$$

is a strict local martingale by our earlier argument (for $m = n-1$). This completes the proof. □

As a final area of applications of h-transforms which are locally strictly dominated, let us mention the theory of measure-valued processes, in particular, the Dawson-Watanabe superprocesses. A well-known example is conditioning a superprocess to survive forever, which can be done by changing the law of a superprocess by using the total mass process (which is a martingale) as the Radon-Nikodým derivative. Please see the seminal article in this direction by Evans & Perkins [10], the book by Etheridge [9], and a follow-up article on similar other h-transforms by Overbeck [26].

3 A converse to the previous result

As a converse to Proposition 1 it turns out that all strict local martingales which remain strictly positive throughout can be obtained as the reciprocal of a martingale under an h-transform. This was essentially proved by Delbaen and Schachermayer [6] in 1995 in their analysis of arbitrage possibilities in Bessel processes. We replicate their theorem below. The construction is related to the Föllmer measure of a positive supermartingale [13].

Before we state the result we need a technique which adds an extra absorbing point *infinity* to the state space $\mathbb{R}^+$, originally inspired by the work of P. A. Meyer [25]. We follow closely the notation used in [6]. The space of trajectories is the space $C_\infty[0,T]$ or $C_\infty[0,\infty)$ of continuous paths ω defined on the time interval $[0,T]$ or $[0,\infty)$ with values in $[0,\infty]$ with the extra property that if $\omega(t)=\infty$, then $\omega(s)=\infty$ for all $s>t$. The topology endowed is the one associated with local uniform convergence. The coordinate process is denoted by X, i.e., $X(t)=\omega(t)$.

Theorem 2 (Delbaen and Schachermayer, Theorem 4 in [6])**.** *If R is a measure on $C[0,1]$ such that X is a strictly positive strict local martingale, then*

(i) there is a probability measure R^* on $C_\infty[0,1]$ such that $M=1/X$ is an R^* martingale.

(ii) We may choose R^* in such a way that the measure R is absolutely continuous with respect to R^* and its Radon-Nikodým derivative is given by $dR=M_1 dR^*$.

The following result is a corollary.

Proposition 6. *Let R be a probability measure on $C[0,\infty)$ under which the coordinate process X_t is a positive strict local martingale starting from one. Then there exists a probability measure Q on the canonical space such that X is a nonnegative martingale under Q and the following holds:*

(i) The probability measure defined by

$$P(A) := E^Q\left(X_t 1_A\right), \quad \forall\, A\in\mathcal{F}_t,\ t\geq 0, \tag{9}$$

is the law of the process $\{1/X_t,\ t\geq 0\}$ under R.

(ii) X is a strict local martingale if and only if $Q(\tau_0<\infty)>0$, where $\tau_0=\inf\{t\geq 0:\ X_t=0\}$ is the first hitting time of zero.

Proof. The only difference in the first part of this proposition with the previous theorem is that the construction is on the entire space $C[0,\infty)$. Note that, by scaling time, Theorem 2 holds for any time interval $[0,T]$, $T = 1, 2, \ldots$. In other words, for every positive integer T, there is a probability measure R^*_T which satisfies the two conditions in Theorem 2 in time interval $[0,T]$. Let Q_T be the law of $1/X$ under R^*_T up to time T. Once we demonstrate that this tower of probability measures is consistent, it follows from standard arguments that they induce a probability measure Q on the entire space $C[0,\infty)$ with the required properties holding locally.

However, consistency is immediate once both the properties (i) and (ii) in Theorem 2 hold for each interval $[0,T]$.

Part (ii) follows from Proposition 1. □

4 Stochastic calculus with strict local martingales

Inspired by the previous representation theorem, we make the following definition:

Definition. We call an ordered pair probability measures (R, Q), defined on the canonical sample space of continuous paths, a Girsanov pair if

1. under R, the coordinate process X_t is a positive strict local martingale starting from one;
2. under Q the process X_t is a nonnegative martingale;
3. The laws R and Q are related by Proposition 6.

The advantage of Proposition 6 is that it allows us to transport stochastic calculus with respect to strict local martingales to that with actual martingales via a change of measure.

Proposition 7. *Let (R, Q) be a Girsanov pair of probability laws. Also let τ_0 denote the hitting time of zero.*

Consider any nonnegative function $h : (0,\infty) \to \mathbb{R}^+$. For any bounded stopping time τ, we get

$$E^R\left(h(X_\tau)\right) = E^Q g(X_\tau) 1_{\{\tau_0 > \tau\}}. \tag{10}$$

Here g is the function $g(x) = xh(1/x)$, for all $x > 0$.

Now suppose $\lim_{x\to 0} g(x) = \eta < \infty$. Define a map $\bar{g} : [0,\infty) \to \mathbb{R}$ by extending g continuously, i.e., $\bar{g}(x) = g(x)$ for $x > 0$, and $\bar{g}(0) = \eta$. Then we have

$$E^R\left(h(X_\tau)\right) = E^Q \bar{g}(X_\tau) - \eta Q\left(\tau_0 \le \tau\right). \tag{11}$$

Before we prove the statement above, as an example note that when $h(x) = (x-a)^+$ for some $a \geq 0$, we get

$$E^R (X_\tau - a)^+ = E^Q (1 - aX_\tau)^+ - Q(\tau_0 \leq \tau). \tag{12}$$

Proof of Proposition 7. This is immediate from the absolute continuity relationship between the laws of the two processes. Note that, by nonnegativity of the martingale M, we have $M_\tau = M_{\tau \wedge \tau_0} = M_\tau 1_{\{\tau_0 > \tau\}}$. One gets

$$E^R h(X_\tau) = E^Q X_\tau h\left(\frac{1}{X_\tau}\right) 1_{\{\tau_0 > \tau\}} = E^Q g(X_\tau) 1_{\{\tau_0 > \tau\}}.$$

For the second assertion assume that one can define $\bar{g}(0) = \eta$ by continuously extending g. Now for any nonnegative path ω which gets absorbed upon hitting zero, the following is an algebraic identity:

$$g(\omega_\tau) 1_{\{\tau_0 > t\}} = g(\omega_\tau) - \eta 1_{\{\tau_0 \leq \tau\}}.$$

In particular this identity holds pathwise when ω is a path of a nonnegative martingale M. Taking expectation on both sides of the last equation with respect to the law of M, we obtain

$$E^Q g(X_\tau) 1_{\{\tau_0 > t\}} = E^Q \bar{g}(X_\tau) - \eta Q(\tau_0 \leq t).$$

This proves the proposition. □

Corollary 1. *Let $h : (0, \infty) \to (0, \infty)$ be a function which is sublinear at infinity, i.e.,*

$$\lim_{x \to \infty} \frac{h(x)}{x} = 0. \tag{13}$$

Then, for all bounded stopping times τ, one has

$$E^R h(X_\tau) = E^Q \bar{g}(X_\tau), \quad \bar{g}(x) = x h(1/x),\ x > 0,\ \bar{g}(0) = 0. \tag{14}$$

Proof. The first part follows directly from (11) since

$$\eta = \lim_{x \to 0} x h\left(\frac{1}{x}\right) = \lim_{x \to \infty} \frac{h(x)}{x} = 0.$$

The second conclusion is obvious. □

The previous corollary has some interesting consequences. For example, when h is convex, it is not difficult to verify that so is $\bar{g}$. And thus both $h(X)$ and $\bar{g}(X)$ are submartingales (under R and Q respectively) by (14). This is in spite of the strictness in the local martingale property of the coordinate process under R. Additionally, if h is symmetric with respect to inverting x, i.e. $h(x) = xh(1/x)$, then $\bar{g} = h$. Hence strictness of local martingales have no effect when these functions are applied. For example, $E^R\sqrt{X_\tau} = E^Q\sqrt{X_\tau}$.

Before we end, let us mention that a more precise result can be obtained from the semimartingale decomposition formulas of Madan and Yor [23].

5 Convex functions of strict local martingales

Strict local martingales are known for odd behavior which is not shared by martingales. For example, a convex function of a martingale is always a submartingale. This need not be the case with local martingales. However if N is a nonnegative strict local martingale, and h is a convex function sublinear at infinity, then $h(N_t)$ is again a submartingale. This is in contrast to functions which are linear at infinity. For example, in the case of $h(x) = x$, the process is actually a supermartingale. Here we demonstrate another example, for the function $(x - K)^+$ with $K > 0$.

5.1 A curious property of the inverse Bessel process

Proposition 8. *Let X_t be a BES(3) process starting from one. For any real $K \in [0, 1/2]$, the function $t \mapsto E\{(1/X_t - K)^+\}$ is strictly decreasing for all $t \in (0, \infty)$. However, if $K > 1/2$, the function $t \mapsto E\{(1/X_t - K)^+\}$ is initially increasing and then strictly decreasing for*

$$t \geq \left(K \log \frac{2K+1}{2K-1} \right)^{-1}.$$

*R*emark: Note that the bound on the right side becomes zero when $K = 1/2$ which demonstrates its sharpness.

Proof of Proposition 8. We use the change of measure relationship (4). Let B be a one-dimensional Brownian motion starting from one and absorbed at zero. We deduce the following identity

$$\begin{aligned} h(t) &:= E\left\{\left(\frac{1}{X_t}-K\right)^+\right\} = E\left\{B_{t\wedge\tau_0}\left(\frac{1}{B_t}-K\right)^+\right\} \\ &= E\left[(1-KB_t)^+ 1_{\{\tau_0>t\}}\right] = E(1-KB_{t\wedge\tau_0})^+ - P(\tau_0\le t). \end{aligned}$$

where τ_0 is the hitting time of zero for the Brownian motion B.

If we take derivatives with respect to t in the equation above, we get

$$\begin{aligned} h'(t) &= \frac{d}{dt}E(1-KB_{t\wedge\tau_0})^+ - \frac{d}{dt}P(\tau_0\le t) \\ &= \frac{K}{2}\frac{d}{dt}EL^{1/K}_{t\wedge\tau_0} - \frac{d}{dt}P(\tau_0\le t). \end{aligned} \tag{15}$$

The second equality above is due to Tanaka formula.

Now the second term on the right side of (15) above is the density of the first hitting time of zero, which we know ([19, page 80]) to be

$$\frac{1}{\sqrt{2\pi t^3}}e^{-1/2t}. \tag{16}$$

To compute the first term on the right of (15) we have the following claim.

Lemma 9. *Suppose $\{X_t,\ t\ge 0\}$ is a continuous nonnegative local martingale which satisfies the following SDE*

$$dX_t = \sigma(t,X_t)d\beta_t, \quad t\in[0,\infty), \quad X_0=1. \tag{17}$$

Here β is a one-dimensional standard Brownian motion and $\sigma(t,x)$ is some measurable nonnegative function on $\mathbb{R}^+\times\mathbb{R}^+$.

Further assume that the process X_t admits a continuous marginal density at each time t at every strict positive point y which is given by

$$p_t(y) = P\left(X_t\in dy \,\middle|\, X_0=1\right), \quad y>0.$$

Let L^a_t denote the local time of X at level $a>0$ and at time t. Then

$$\frac{d}{dt}E\left(L^a_t\right) = \sigma^2(t,a)p_t(a). \tag{18}$$

Using the previous lemma, we can explicitly compute the right side of equation (15). Recall (see [19, page 97]) that for x, y, and t strictly positive the transition function of Brownian motion absorbed at zero is given by

$$p(t,x,y) := \frac{1}{\sqrt{2\pi t}}\left[\exp\left(-\frac{(y-x)^2}{2t}\right) - \exp\left(-\frac{(y+x)^2}{2t}\right)\right].$$

Thus, combining (15), (16), and (18), we get

$$h'(t) = \frac{K}{2\sqrt{2\pi t}}\left[e^{-(1-1/K)^2/2t} - e^{-(1+1/K)^2/2t}\right] - \frac{1}{\sqrt{2\pi t^3}}e^{-1/2t}. \tag{19}$$

Thus, $h'(t) < 0$ if and only of

$$\begin{aligned}\frac{2}{Kt} &> e^{1/2t}\left[e^{-(1-1/K)^2/2t} - e^{-(1+1/K)^2/2t}\right]\\ &= \exp\left[\frac{(2K-1)}{2K^2t}\right] - \exp\left[-\frac{(2K+1)}{2K^2t}\right].\end{aligned} \tag{20}$$

We need to do a bit more work. Let $t = 1/y$. Consider the function on the right side of the last inequality. We need to consider two separate cases. First suppose $K > 1/2$. Then both $2K-1$ and $2K+1$ are positive. If for two positive parameters $\lambda_2 > \lambda_1 > 0$, we define a function q by $q(y) = \exp(\lambda_1 y) - \exp(-\lambda_2 y)$, $y > 0$, it then follows that

$$\begin{aligned}q'(y) &= \lambda_1 e^{\lambda_1 y} + \lambda_2 e^{-\lambda_2 y}, \quad q'(0) = \lambda_1 + \lambda_2,\\ q''(y) &= \lambda_1^2 e^{\lambda_1 y} - \lambda_2^2 e^{-\lambda_2 y}.\end{aligned} \tag{21}$$

Note that $q''(y) < 0$, for all

$$0 \le y < \frac{2\log(\lambda_2/\lambda_1)}{\lambda_1+\lambda_2}. \tag{22}$$

Since $q'(y)$ is always positive, it follows that q is an increasing concave function starting from zero in the interval given by (22). Thus it also follows that, in that interval,

$$q(y) = q(y) - q(0) < yq'(0) = y(\lambda_1+\lambda_2). \tag{23}$$

Take $\lambda_1 = (2K-1)/2K^2$ and $\lambda_2 = (2K+1)/2K^2$. Then $\lambda_1+\lambda_2 = 2/K$. By (22) we get that if

$$y \le C_1 := K\log\frac{2K+1}{2K-1},$$

then, from (23) it follows

$$K\left\{\exp\left[\frac{(2K-1)y}{2K^2}\right]-\exp\left[-\frac{(2K+1)y}{2K^2}\right]\right\}<2y.$$

That is, by (20), $h'(t)<0$, i.e., h is strictly decreasing for all

$$t>\left(K\log\frac{2K+1}{2K-1}\right)^{-1}.$$

The case when $0<K\le 1/2$ can handled similarly. Suppose $0<\lambda_1<\lambda_2$ are positive constants. Consider the function

$$r(y)=-\lambda_1 y+\lambda_2 y-e^{-\lambda_1 y}+e^{-\lambda_2 y},\quad y\in[0,\infty).$$

Then $r(0)=0$, and

$$r'(y)=-\lambda_1\left(1-e^{-\lambda_1 y}\right)+\lambda_2\left(1-e^{-\lambda_2 y}\right)>0,\quad y\in[0,\infty),$$

because $\lambda_1<\lambda_2$. Thus, for all positive y, we have $r(y)>0$, i.e.,

$$e^{-\lambda_1 y}-e^{-\lambda_2 y}<(-\lambda_1+\lambda_2)y.$$

We use this for $\lambda_1=(1-2K)/2K^2$ and $\lambda_2=(1+2K)/2K^2$. Note that, as before

$$(-\lambda_1+\lambda_2)y=2y/K.$$

From (20) it follows that $h'(t)>0$ for all $t\in(0,\infty)$. Thus we have established that if $K\le 1/2$, the function $t\mapsto E(1/X_t-K)^+$ is strictly decreasing for all $t\in(0,\infty)$. This completes the proof of the proposition. □

Proof of Lemma 9. Several conditions for the existence and uniqueness of such a one-dimensional equation which does not explode can be found in the literature. For example, it is sufficient to have Lipschitz continuity in space, and joint measurability (see, for example [29, Chapter V, Section 3]).

To prove this, we use the occupation time formula involving the local time for general continuous semimartingales.

For any smooth nonnegative function $f:\mathbb{R}\to\mathbb{R}^+$ with compact support contained in $(0,\infty)$, we have the following identity

$$\int_{\mathbb{R}^+} f(a)L_t^a da=\int_0^t f(X_s)d\langle X\rangle_s=\int_0^t f(X_s)\sigma^2(s,X_s)ds,$$

where the final identity follows from (17). Now taking expectations on both sides, we obtain

$$E\left[\int_{\mathbb{R}^+} f(a)L_t^a da\right] = E\int_0^t f(X_s)\sigma^2(s,X_s)ds = \int_0^t E\left[f(X_s)\sigma^2(s,X_s)\right]ds$$
$$= \int_0^t \left[\int_{\mathbb{R}^+} f(a)\sigma^2(s,a)p_s(a)da\right]ds. \tag{24}$$

The second equality above is due to Fubini-Tonelli for nonnegative integrands. The final equality is by definition of the marginal density and the fact that the support of f is in $(0,\infty)$.

Applying Fubini-Tonelli repeatedly and interchanging the orders of integration on both sides of (24), we get

$$\int_{\mathbb{R}^+} f(a)E(L_t^a)da = E\left[\int_{\mathbb{R}^+} f(a)L_t^a da\right] = \int_0^t \left[\int_{\mathbb{R}^+} f(a)\sigma^2(s,a)p_s(a)da\right]ds$$
$$= \int_{\mathbb{R}^+} f(a)\left[\int_0^t \sigma^2(s,a)p_s(a)ds\right]da.$$

Since this holds for all smooth nonnegative functions f with compact support in $(0,\infty)$, it follows that

$$E(L_t^a) = \int_0^t \sigma^2(s,a)p_s(a)ds, \quad \forall\, a>0.$$

The conclusion of the lemma follows.

$\square$

For mathematical completeness we show below that a similar result can be proved for the Bessel process starting from zero, although in this case there is no dependence on K. The proof is much simpler and essentially follows by a scaling argument. Note that, even in this case the reciprocal of the Bessel process is well-defined for all times except at time zero. Hence $1/X_t$, $t\in(0,\infty)$, can be thought as a Markov process with an *entrance distribution*, i.e., a pair consisting of a time-homogenous Markov transition kernel $\{P_t\}$, $t>0$, and a family of probability measures $\{\mu_s\}$, $s>0$, satisfying the constraint $\mu_s * P_t = P_{t+s}$. Here $*$ refers to the action of the kernel on the measure.

Proposition 10. *Let X_t be a 3-dimensional Bessel process, BES(3), such that $X_0=0$. For any two time points $u>t>0$, and for $K\geq 0$, one has*

$$E\left(\frac{1}{X_u}-K\right)^+ < E\left(\frac{1}{X_t}-K\right)^+. \tag{25}$$

Proof. Fix $u > t$. Recall that BES(3), being the norm of a three dimensional Brownian motion, has the Brownian scaling property when starting from zero. That is to say, for any $c > 0$,

$$\left(\frac{1}{\sqrt{c}} X_{cs},\ s \geq 0 \right) \stackrel{\mathcal{L}}{=} (\ X_s,\ s \geq 0\ ),$$

where the above equality is equality in law.

Take $c = u/t$, and apply the above equality for X_s when $s = t$, to infer that $c^{-1/2} X_u$ has the same law as X_t, and thus

$$E\left(\frac{1}{X_u} - K\right)^+ = E\left(\frac{c^{-1/2}}{X_t} - K\right)^+ = c^{-1/2} E\left(\frac{1}{X_t} - \sqrt{c}K\right)^+ . \qquad (26)$$

Note that for any $\sigma > 1$, we have $(x - \sigma K)^+ / \sigma < (x - K)^+, \quad \forall\, x > 0$. Since $c > 1$, taking $\sigma = \sqrt{c}$, one deduces from (26)

$$E\left(\frac{1}{X_u} - K\right)^+ < E\left(\frac{1}{X_t} - K\right)^+,$$

which proves the result. □

We conclude this subsection with an example of a strict local martingale S where $E(S_t - K)^+$ is not asymptotically decreasing for any K. This, coupled with the earlier Bessel result, establishes the fact that functions which are not sublinear at infinity can display a variety of characteristics when applied to strict local martingales.

We inductively construct a process in successive intervals $[i, i+1)$ by the following recipe. The process starts at zero. The process in the odd interval $[2i, 2i+1)$ is an exponential Brownian motion $\exp(B_t - t/2)$ starting from S_{2i} and independent of the past. On the even intervals $[2i+1, 2i+2)$ the process S is an inverse Bessel process starting from S_{2i+1} and again independent of the past. The constructed process is always a positive local martingale. The value of the function $E(S_t - K)^+$ is increasing in the odd intervals due to the martingale component, and decreasing (at least when $K \leq 1/2$) in the Bessel component by Proposition 8.

One might object to the fact that this process is not strict local throughout. But, one can mix the two components, by a sequence of coin tosses which decides whether to use Brownian or the Bessel component in the corresponding interval. By choosing the probability of heads in these coins in a suitably predictable manner, we can generate a local martingale which is strict throughout but $E(S_t - K)^+$ does not decrease anywhere.

5.2 A multidimensional analogue by Kelvin transform

In the last section we saw that for any strict local martingale law R there is a true martingale law Q such that equality (10) holds. The transformation $g(x) = xh(1/x)$ has a well-known analogue in dimensions higher than two called the Kelvin transform. In our final subsection we present an interesting multidimensional generalization of our results for dimensions $d > 2$. We take the following definition from the excellent book on harmonic function theory [1, Chapter 4] by Axler, Bourdon, and Ramey.

The Kelvin transform K is an operator acting on the space of real functions u on a subset of $\mathbb{R}^d \backslash \{0\}$. Let u be a C^2 function on an open subset $\mathcal{D}$ of $\mathbb{R}^d \backslash \{0\}$. Let $\mathcal{D}^*$ be the image of $\mathcal{D}$ under the inversion map

$$x \mapsto x^* = x / |x|^2 .$$

For such a u, we define a function $K[u] : \mathcal{D}^* \to \mathbb{R}$ by the formula

$$K[u](y) = |y|^{2-d} u \left(y/|y|^2 \right) . \tag{27}$$

Notice that K is its own inverse.

The most striking property of this transform is that K *commutes* with the Laplacian ([1, page 62]). Let v be the function $v(x) = |x|^4 \Delta u(x),\ x \in \mathcal{D}$. Then, at any point $y \in \mathcal{D}^*$, we have

$$\Delta K[u](y) = K[v](y). \tag{28}$$

In particular, if u is harmonic in $\mathcal{D}$ (i.e., $\Delta u = 0$), then $K[u]$ is harmonic in $\mathcal{D}^*$. Also, if u is subharmonic (i.e., $\Delta u \geq 0$), then so is $K[u]$.

Now, in one dimension, every local martingale is a time-changed Brownian motion. This can be generalized in higher dimensions by considering a d-dimensional conformal local martingale, i.e., a process $(X_1, \ldots, X_n)$ such that each coordinate X_i is a local martingale and

$$\langle X_i, X_j \rangle = \langle X_1 \rangle 1_{\{i=j\}}, \quad \text{for all } 1 \leq i, j \leq d.$$

We consider it to be *strictly local* if at least one of its coordinate processes is a strict local martingale. We replace the condition of nonnegativity of one dimensional processes by restricting our multidimensional processes in the complement of a compact neighborhood of zero.

Proposition 11. *Let D be a compact neighborhood of the origin and denote its complement $\mathbb{R}^d \backslash D$ by D^c. Let P be a probability measure on the canonical* *sample space such that, under P, the coordinate process X is a conformal local martingale in $\mathbb{R}^d$ ($X_0 = x_0 \in D^c$) absorbed upon hitting the boundary of D. Then there is a probability measure Q such that, under Q, the coordinate process is a true conformal martingale which takes values in D^*. Moreover, for any function $U : D^c \to \mathbb{R}$, and for any bounded stopping time τ, we have*

$$|x_0|^{2-d} E^P \left[U \left(X_\tau \right) \right] = E^Q \left[|X_\tau|^{2-d} U \left(\frac{X_\tau}{|X_\tau|^2} \right) \right]. \tag{29}$$

The proof requires the following lemmas.

Lemma 12. *Let $|\cdot|$ denote the Euclidean norm in dimension d. Let τ_1 be the hitting time of D i.e.,*

$$\tau_1 = \inf \{ t \geq 0 : \ X_t \in D \} .$$

Then, the process $|X_{t\wedge\tau_1}|^{2-d}$, $t \geq 0$, is a martingale.

Proof. The function $|x|^{2-d}$ is harmonic in $\mathbb{R}^d$. Thus $|X_t|^{2-d}$ is a local martingale itself. Since it is bounded in D^c, it must be a true martingale. $\square$

Suppose $X_0 = x_0 \in D^c$. We can change the law of X by using $|X_{t\wedge\tau_1}|^{2-d}$ as a Radon-Nikodým derivative (after normalizing). We get the following lemma.

Lemma 13. *Suppose $X_0 = x_0$ such that $x_0 \in D^c$. We change P by using the positive martingale*

$$\phi(X_t) = |X_{t\wedge\tau_1}|^{2-d} / |x_0|^{2-d}$$

as a Radon-Nikodým derivative. Call this measure Q^*. Under Q^*, the process

$$Y_t = \frac{X_{t\wedge\tau_1}}{|X_{t\wedge\tau_1}|^2} \tag{30}$$

is again a d-dimensional conformal local martingale such that every coordinate process Y is a true martingale.

Proof of Lemma 13. Note that, since D is a neighborhood of zero, the set $(D^c)^*$ is compact. Thus, Y lives in a bounded set. To show that the ith coordinate process $Y(i)$ is a local martingale, we use the harmonic function

$$u(x) = x_i \quad \text{on} \quad \Omega = \mathbb{R}^n \backslash \{0\}.$$

By (28), its Kelvin transform is also harmonic. Hence the process

$$|X_t|^{2-d} u\left(X_t / |X_t|^2\right) = |X_t|^{-d} X_t(i)$$

is a local martingale under P. But, since $|X_t|^{2-d}$ (until τ_1) is used as a Radon-Nikodým derivative after being scaled, by Bayes rule [19, page 193], the process $u\left(X_{t\wedge\tau_1} / |X_{t\wedge\tau_1}|^2\right)$ is a local martingale under the changed measure Q^*. But this implies that $Y(i)$ is a local martingale under Q^*. But, since every $Y(i)$ is bounded they must be true martingales.

To show that Y is conformal, we use the harmonic function $u(x) = x_i x_j$ again on the full domain $\mathbb{R}^n \backslash \{0\}$ for any pair of coordinates $i \neq j$. Exactly as in the previous paragraph, we infer from (28) that $Y_t(i)Y_t(j)$ is a local martingale under Q. But this implies $\langle Y(i), Y(j)\rangle \equiv 0$. That their quadratic variations must be the same follows from symmetry. □

Proof of Proposition 11. The construction of Q has been done in Lemma 13 where it is the law of the process Y under Q^*. Note that the Randon-Nikodým derivative $\phi(X_t)$ never hits zero. Thus, even under Q^*, the process Y never hits zero. To show the equality (29), we use the change of measure to get

$$E^{Q^*}\left[|Y_\tau|^{2-d} U\left(\frac{Y_\tau}{|Y_\tau|^2}\right)\right] = |x_0|^{d-2} E^P\left[|X_\tau|^{2-d} |X_\tau|^{d-2} U\left(X_\tau\right)\right],$$

which is equal to $|x_0|^{d-1} E^P\left[U(X_\tau)\right]$. This proves the result. □

6 Applications to financial bubbles

A natural question is: what happens to a financial market when the no arbitrage condition yields a strict local martingale (rather than a true martingale) under a risk neutral measure? Several authors have looked at this problem and offered solutions to anomalies which might result from the lack of the martingale property. One interesting perspective offered in this direction is the theory of price bubbles as argued in 2000 by Loewenstein and Willard [21]. They propose that to identify a bubble one needs to look at the difference between the market price of an asset and its fundamental price. Their argument is later complemented and further developed by Cox and Hobson [4] and the two articles by Jarrow, Protter, and Shimbo [16], [17]. Please see the latter articles for the definitions of the market and the fundamental prices of an asset and any of the other financial terms that follow. In particular, the authors in [16] and [17] classify bubbles into three types in an arbitrage-free market satisfying Merton's *No Dominance* condition (see [16] or [24]). One, in which the difference between the two price processes under an equivalent local martingale measure is a uniformly integrable martingale; two, when it is a martingale but non-uniformly integrable; and last, when it is a strict local martingale. In a static market with infinite horizon, for a stock which pays no dividends, Example 5.4 in [17] shows that the difference between the two prices is actually the current market price of the stock. Thus a stock price which behaves as a strict local martingale under an equivalent local martingale measure is an example of a price bubble of the third kind. Cox and Hobson [4], too, use this definition of stock price bubbles. They further furnish several interesting examples of bubbles both where volatility increases with price levels, and where the bubble is the result of a feedback mechanism. They go on to exhibit (among other things) how in the presence of bubbles put-call parity might not hold and call prices do not tend to zero as strike tends to infinity.

We consider a market with a single risky asset (stock) and zero spot interest rate. Let $\{S_t\}$, $t \in (0, \infty)$, be a positive continuous strict local martingale which models the discounted price of the (non-dividend paying) stock under an equivalent local martingale measure. We have the following result which follows immediately from Corollary 1 and the subsequent Bessel example.

Proposition 14. *Suppose for a European option, the discounted pay-off at time T is given by a convex function $h(S_T)$ which is sublinear at infinity, i.e., $\lim_{x\to\infty} h(x)/x = 0$. Then the price of the option is increasing with the time to maturity, T, whether or not a bubble is present in the market. In other words, $E(h(S_T))$ is an increasing function of T. For example, consider the put option with a pay-off $(K - x)^+$.*

However, for a European call option, the price of the option $E(S_T - K)^+$ with strike K might decrease as the maturity increases.

This feature may seem strange at first glance, but if we assume the existence of a financial bubble, the intuition is that it is advantageous to purchase a call with a short expiration time, since at the beginning of a bubble prices rise, sometimes dramatically. However in the long run it is disadvantageous to have a call, increasingly so as time increases, since the likelihood of a crash in the bubble taking place increases with time.

Of course, pricing a European option by the usual formula when the underlying asset price is a strict local martingale is itself controversial. For example, Heston, Loewenstein, and Willard [14] observe that under the existence of bubbles in the underlying price process, put-call parity might not hold, American calls have no optimal exercise policy, and lookback calls have infinite value. Madan and Yor [23] have recently argued that when the underlying price process is a strict local martingale, the price of a European call option with strike rate K should be modified as $\lim_{n\to\infty} E\left[(S_{T\wedge T_n} - K)^+\right]$, where $T_n = \inf\{t \geq 0 : S_t \geq n\}$, $n \in \mathbb{N}$, is a sequence of hitting times. This proposal does however, in effect, try to hide the presence of a bubble and act as if the price process is a true martingale under the risk neutral measure, rather than a strict local martingale.

Let us also mention that a different approach to such market anomalies has been studied extensively in Fernholz and Karatzas [11], Fernholz, Karatzas, and Kardaras [12], and Karatzas and Kardaras [18]. In [11] the authors investigate the case when the *candidate Radon-Nikodým derivative* for the risk-neutral measure turns out to be a strict local martingale. See Proposition 3.4 (also Remark 4.2) for the details. This is intimately connected with what the authors call a *weakly diverse market* which results in a number of anomalies similar to the case of bubbles. For example, put-call parity fails to hold in such markets. See, Remark 9.1 and 9.3 in [12]. Also see Example 9.2 for anomalies in the price of European call option.

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