Bubbles and futures contracts in markets with short-selling constraints

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Motivation

- The **current financial crisis** is, to a large extent, a product of the burst of the alleged real estate bubble.
- **Massive short-selling** after the burst of a financial bubble.
- **Short-selling ban, September 2008**: U.S. Securities and Exchange Commission (SEC) and U.K. Financial Services Authority (FSA).
- In most of the **third world emerging markets** the practice of short-selling is not allowed.

Local martingale approach to bubbles

- **Jarrow, Protter and Shimbo (2006, 2008) and Cox and Hobson (2005)**:

$$\boxed{\begin{array}{c}(NFLVR)\\ \text{strategies with bounded liabilities}\end{array}} \Leftrightarrow \boxed{\mathcal{M}_{loc}(S) \neq \emptyset}$$

$$\boxed{\text{Valuation measure } Q^* \in \mathcal{M}_{loc}(S) \backslash \mathcal{M}_{mar}(S)} \Rightarrow \boxed{\text{Bubbles}}$$

$$\boxed{E^{Q^*}[S_T] < S_0}$$

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The model

- **Reference filtered probability space**: $(\Omega, \mathcal{F}, \mathbb{F}, P)$.
- **Price process**: $(S_t)_{0 \le t \le T}$ a nonnegative locally bounded semi-martingale.
- **Money market account:** $R_t \equiv 1$.
- **The admissible strategies**:

$$\mathcal{A} := \left\{ H \in L(S) : H_0 = 0,\ H \ge 0,\ \int_0^{\cdot} H \cdot S \ge -\alpha, \text{ for some } \alpha > 0 \right\}.$$

- **Payoffs of zero initial value portfolios:**

$$\mathcal{K} := \left\{ \int_0^T H_s \, dS_s : H \in \mathcal{A} \right\} \subset L^0(\Omega, \mathcal{F}, P).$$

- **Bounded payoffs dominated by elements of $\mathcal{K}$:**

$$\mathcal{C} := (\mathcal{K} - L^0_+(\Omega, \mathcal{F}, P)) \cap L^\infty(\Omega, \mathcal{F}, P) \subset L^\infty(\Omega, \mathcal{F}, P).$$

The FTAP under short-selling prohibition

Theorem

Let $\mathcal{M}_{sup}(S)$ be the set of probability measures $Q \sim P$ such that S is a Q-supermartingale. Then

$$\text{(NFLVR)} \Leftrightarrow \overline{\mathcal{C}} \cap L^{\infty}_{+}(\Omega, \mathcal{F}, P) = \{0\} \Leftrightarrow \mathcal{M}_{sup}(S) \neq \emptyset.$$

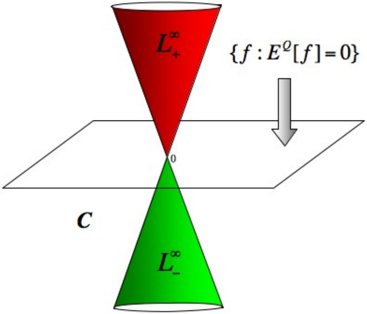
Related results:

- L^2 case for simple strategies: Jouini and Kallal (1995).
- Simple predictable strategies in L^{∞}: Frittelli (1997).

A key observation

Proposition (Extension of Ansel and Stricker, 1994)

$$\mathcal{M}_{sup}(S) = \left\{ Q \sim P : \int_0^{\cdot} H\, dS \textit{ is a } Q\textit{-supermartingale for all } H \in \mathcal{A} \right\}.$$



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Replication under short selling prohibition

Theorem (Extension of Ansel and Stricker, 1994)

Suppose $\mathcal{M}_{sup}(S) \neq \emptyset$. For $f_T \in L^0_+(\Omega, \mathcal{F}, P)$ TFAE

(i) $f_T = x + \int_0^T H_s\, dS_s$ *with x constant and $H \in \mathcal{A}$ such that* $\int_0^\cdot H_s\, dS_s$ *is a Q^*-martingale for some $Q^* \in \mathcal{M}_{sup}(S)$.*

(ii) *There exists $Q^* \in \mathcal{M}_{sup}(S)$ such that*

$$\sup_{Q \in \mathcal{M}_{sup}(S)} E^Q[f_T] = E^{Q^*}[f_T] < \infty$$

This theorem is a corollary of a more general result proved by Föllmer & Kramkov (1997).

Example

If the price process is continuous (and nonconstant) the payoff $f_T = 1_{(S_T < S_0)}$ cannot be perfectly replicated without short-selling.

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Futures contracts

Purchase of S at time T via prearranged payment procedure.
Importance: **(1) Cash-flow depends on market valuation (2) Very liquid derivatives.**

Definition (Karatzas and Shreve, Methods of Mathematical Finance)

A futures contract on S with maturity time T is a financial instrument with associated stream of cash-flows $F_{t,T}$, such that

(i) *$F_{t,T}$ is a nonnegative $\mathbb{F}$-adapted semi-martingale with $F_{T,T} = S_T$.*
(ii) *The market price of the stream of cash-flows $(F_{t,T})_t$ is zero at all times.*

$F_{t,T}$ is known as the futures price process.

If this contract can be sold short, in the extended market

$$\text{(NFLVR)} \Leftrightarrow \boxed{F_{t,T} \text{ is a } Q\text{-local martingale for some } Q \in \mathcal{M}_{sup}(S)}$$

Completing with futures

Theorem (Completing with futures - No interest rates)

Suppose that S is positive and continuous, $\mathcal{M}_{loc}(S) = \{P\}$ and $Q \in \mathcal{M}_{sup}(S)$. If

$$S = M - A = \mathcal{E}(-B)\mathcal{E}(N),$$

with $M, \mathcal{E}(N)$ Q-martingales and A, B increasing,

$$F_{t,T} = E^Q[S_T|\mathcal{F}_t].$$

and B is deterministic then $\mathcal{M}_{loc}(F_{\cdot,T}) = \{Q\}$.

Lemma

Suppose that S is positive and continuous, $\mathcal{M}_{loc}(S) = \{P\}$, $Q \in \mathcal{M}_{sup}(S)$ and $S = M - A$ is the Doob-Meyer decomposition of S under Q. Then

$$\mathcal{M}_{loc}(M) = \{Q\}.$$

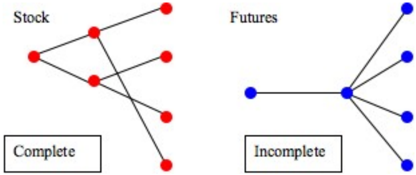
Pathological examples

- Cox and Hobson 2005

$$S = 1 + \mathcal{E}\left(\int_0^{\cdot} \frac{dB_s}{\sqrt{T-s}}\right).$$

Strict local martingale $S_T \equiv 1$, hence $F_{t,T} \equiv 1$.

- Binary tree



Open question

Suppose that $|\mathcal{M}_{loc}(S)| = 1$.

- Find necessary and sufficient conditions on $Q \in \mathcal{M}_{sup}(S)$ under which the futures (+ bonds) market is complete.

Thank you!
Questions?

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Current work and open questions

- Minimal entropy and minimal variance super-martingale measures.
- Specific models analysis: Stochastic volatility, models with jumps.
- More general conditions on Q to assure completeness.
- Liquidity aspects

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