Bubbles and futures contracts in markets with short-selling constraints

Sergio Pulido, Cornell University
PhD committee: Philip Protter, Robert Jarrow

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Motivation

- The **current financial crisis** is, to a large extent, a product of the burst of the alleged real estate bubble.

- **Massive short-selling** after the burst of a financial bubble.

- **Short-selling ban, September 2008**: U.S. Securities and Exchange Commission (SEC) and U.K. Financial Services Authority (FSA).

- In most of the **third world emerging markets** the practice of short-selling is not allowed.
Local martingale approach to bubbles

- Jarrow, Protter and Shimbo (2006, 2008) and Cox and Hobson (2005):

\[
\text{(NFLVR)} \quad \iff \quad \mathcal{M}_{loc}(S) \neq \emptyset
\]

Valuation measure \( Q^* \in \mathcal{M}_{loc}(S) \setminus \mathcal{M}_{mar}(S) \) \( \Rightarrow \) Bubbles

\[ E^{Q^*}[S_T] < S_0 \]
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The model

- **Reference filtered probability space:** \((\Omega, \mathcal{F}, \mathbb{F}, P)\).
- **Price process:** \((S_t)_{0 \leq t \leq T}\) a nonnegative locally bounded semi-martingale.
- **Money market account:** \(R_t \equiv 1\).
- **The admissible strategies:**
  \[ A := \left\{ H \in L(S) : H_0 = 0, \ H \geq 0, \ \int_0^T H \cdot S \geq -\alpha, \text{ for some } \alpha > 0 \right\}. \]
- **Payoffs of zero initial value portfolios:**
  \[ \mathcal{K} := \left\{ \int_0^T H_s \, dS_s : H \in A \right\} \subset L^0(\Omega, \mathcal{F}, P). \]
- **Bounded payoffs dominated by elements of \(\mathcal{K}\):**
  \[ C := (\mathcal{K} - L^0_+(\Omega, \mathcal{F}, P)) \cap L^\infty(\Omega, \mathcal{F}, P) \subset L^\infty(\Omega, \mathcal{F}, P). \]
The FTAP under short-selling prohibition

**Theorem**

Let $\mathcal{M}_{sup}(S)$ be the set of probability measures $Q \sim P$ such that $S$ is a $Q$-supermartingale. Then

\[
(\text{NFLVR}) \iff \overline{C} \cap L^\infty(\Omega, \mathcal{F}, P) = \{0\} \iff \mathcal{M}_{sup}(S) \neq \emptyset.
\]

**Related results:**

A key observation

Proposition (Extension of Ansel and Stricker, 1994)

\[ \mathcal{M}_{\text{sup}}(S) = \left\{ Q \sim P : \int_0^\cdot H \, dS \text{ is a } Q\text{-supermartingale for all } H \in \mathcal{A} \right\} . \]
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Replication under short selling prohibition

Theorem (Extension of Ansel and Stricker, 1994)

Suppose $\mathcal{M}_{sup}(S) \neq \emptyset$. For $f_T \in L^0_+ (\Omega, \mathcal{F}, P)$ TFAE

(i) $f_T = x + \int_0^T H_s \, dS_s$ with $x$ constant and $H \in \mathcal{A}$ such that
$\int_0^T H_s \, dS_s$ is a $Q^*$-martingale for some $Q^* \in \mathcal{M}_{sup}(S)$.

(ii) There exists $Q^* \in \mathcal{M}_{sup}(S)$ such that

$$\sup_{Q \in \mathcal{M}_{sup}(S)} E^Q [f_T] = E^{Q^*} [f_T] < \infty$$

This theorem is a corollary of a more general result proved by Föllmer & Kramkov (1997).

Example

If the price process is continuous (and nonconstant) the payoff $f_T = 1_{(S_T < S_0)}$ cannot be perfectly replicated without short-selling.
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Futures contracts

Purchase of $S$ at time $T$ via prearranged payment procedure.

Importance: **(1) Cash-flow depends on market valuation (2) Very liquid derivatives.**

**Definition (Karatzas and Shreve, Methods of Mathematical Finance)**

A futures contract on $S$ with maturity time $T$ is a financial instrument with associated stream of cash-flows $F_{t,T}$, such that

(i) $F_{t,T}$ is a nonnegative $\mathbb{F}$-adapted semi-martingale with $F_{T,T} = S_T$.

(ii) The market price of the stream of cash-flows $(F_{t,T})_t$ is zero at all times.

$F_{t,T}$ is known as the futures price process.

If this contract can be sold short, in the extended market

$$(NFLVR) \iff F_{t,T} \text{ is a } Q\text{-local martingale for some } Q \in \mathcal{M}_{\sup}(S)$$
Completing with futures

Theorem (Completing with futures - No interest rates)

Suppose that $S$ is positive and continuous, $\mathcal{M}_{loc}(S) = \{P\}$ and $Q \in \mathcal{M}_{sup}(S)$. If

$$S = M - A = \mathcal{E}(-B)\mathcal{E}(N),$$

with $M, \mathcal{E}(N)$ $Q$-martingales and $A, B$ increasing,

$$F_{t,T} = E^Q[S_T|\mathcal{F}_t].$$

and $B$ is deterministic then $\mathcal{M}_{loc}(F_{t,T}) = \{Q\}$.

Lemma

Suppose that $S$ is positive and continuous, $\mathcal{M}_{loc}(S) = \{P\}$, $Q \in \mathcal{M}_{sup}(S)$ and $S = M - A$ is the Doob-Meyer decomposition of $S$ under $Q$. Then

$$\mathcal{M}_{loc}(M) = \{Q\}.$$
Pathological examples

- Cox and Hobson 2005

\[ S = 1 + \mathcal{E} \left( \int_0^T \frac{dB_s}{\sqrt{T-s}} \right). \]

Strict local martingale \( S_T \equiv 1 \), hence \( F_{t,T} \equiv 1 \).

- Binary tree
Open question

Suppose that $|\mathcal{M}_{loc}(S)| = 1$.

- Find necessary and sufficient conditions on $Q \in \mathcal{M}_{sup}(S)$ under which the futures (+ bonds) market is complete.
Thank you!
Questions?
References

Current work and open questions

- Minimal entropy and minimal variance super-martingale measures.
- Specific models analysis: Stochastic volatility, models with jumps.
- More general conditions on $Q$ to assure completeness.
- Liquidity aspects

Air China Ltd: Shanghai Vs Hong Kong

![Chart showing stock price movements of Air China Ltd in Shanghai and Hong Kong markets from 2007 to 2009]

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