Optimal trading strategies under arbitrage

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How should an investor trade and how much capital does she need?

- Imagine an investor who wants to hold the stock S_i with price $S_i(0)$ of a company in a year.
- Surely, she could just buy the stock today for a price $S_i(0)$.
- This might not be an "optimal strategy", even under a classical no-arbitrage situation ("no free lunch with vanishing risk").
- There can be other "strategies" which require less initial capital than $S_i(0)$ but enable her to hold the stock after one year.
- But how much initial capital does she need at least and how should she trade?

Two examples where the stock price does not reflect the hedging price.

- Reciprocal of the three-dimensional Bessel process (NFLVR):

$$d\tilde{S}(t) = -\tilde{S}^2(t)dW(t)$$

- Three-dimensional Bessel process:

$$dS(t) = \frac{1}{S(t)}dt + dW(t)$$

How can a portfolio optimally been hedged?

- Optimality in the sense of smallest initial capital.
- Solved by Fernholz D. & Karatzas I. for the market portfolio.
- "Hedging price" characterized via a PDE which can allow for non-unique solutions.
- Change of measure to simplify computations.
- In Markovian framework.
- No martingale representation theorem used.

We assume a Markovian market model.

- Our time is finite: $T < \infty$. Interest rates are zero.
- The stocks $S(\cdot) = (S_1(\cdot), \dots, S_n(\cdot))^{\mathcal{T}}$ follow

$$dS_i(t) = S_i(t) \left(\beta_i(t, S(t))dt + \sum_{k=1}^{K} \sigma_{i,k}(t, S(t))dW_k(t) \right)$$

  with some measurability and integrability conditions.
- $\rightarrow$ Markovian
- but not necessarily complete ($K > n$ allowed).
- The covariance process is defined as

$$\alpha_{i,j}(t, S(t)) := \sum_{k=1}^{K} \sigma_{i,k}(t, S(t))\sigma_{j,k}(t, S(t))$$

Two important guys: the market price of risk and its close relative, the stochastic discount factor.

- The market price of risk is the function ν satisfying

$$\mathbb{P}\left(\{\beta = \sigma\nu \forall t \in [0;T]\} \bigcap \left\{\int_0^T \|\nu\|^2 dt < \infty\right\}\right) = 1.$$

- The market price of risk is not necessarily unique.
- Related is the stochastic discount factor

$$Z^\nu(t) := \exp\left(-\int_0^t \nu^\mathcal{T}(u, S(u))dW(u) - \frac{1}{2}\int_0^t \|\nu(u, S(u))\|^2 du\right)$$

with dynamics

$$dZ^\nu(t) = -\nu^\mathcal{T}(t, S(t))Z^\nu(t)dW(t)$$

- If Z^ν is a strict local martingale, arbitrage is possible.

Everything an investor cares about: how and how much?

- We focus on Markovian trading strategies $\pi(t, S(t))$.
- Which percentage of the total wealth is invested in the single stocks depends on time and current market situation.
- The corresponding wealth process $V^{v,\pi}$ follows

$$\frac{dV^{v,\pi}(t)}{V^{v,\pi}(t)} = \sum_{i=1}^{n} \pi_i(t, S(t)) \frac{dS_i(t)}{S_i(t)}$$

  and is usually not Markovian.
- We are ready to define the π-specific price of risk as

$$\nu^{\pi}(t, s) := \nu(t, s) - \sigma^{\mathcal{T}}(t, s)\pi(t, s).$$

- Then

$$\frac{d(V^{v,\pi}(t)Z^{\nu}(t))}{V^{v,\pi}(t)Z^{\nu}(t)} = -\sum_{k=1}^{K} \nu_k^{\pi}(t, S(t))dW_k(t).$$

We explicitly allow for arbitrage.

It can be shown that excluding arbitrage a priori leads to market models which cannot capture important properties of the real markets very well. (diversity, behavior of market weights)

- We call a strategy ρ with $\mathbb{P}(V^{\rho}(T) \geq V^{\pi}(T)) = 1$ and $\mathbb{P}(V^{\rho}(T) > V^{\pi}(T)) > 0$ relative arbitrage opportunity with respect to the strategy π.
- We call ρ a classical arbitrage opportunity if π invests fully in the bond, that is, if $\pi(t, s) \equiv 0$ for all $(t, s) \in [0; T] \times \mathbb{R}_{+}^{n}$.

But we exclude some forms of arbitrage.

- Remember: We have assumed that there exists some ν which maps the volatility into the drift, that is $\sigma(\cdot,\cdot)\nu(\cdot,\cdot) = \beta(\cdot,\cdot)$.
- It can be shown that this assumption excludes "unbounded profit with bounded risk".
- Thus "making (a considerable) something out of almost nothing" is not possible.
- However, it is still possible to "certainly make something more out of something".

A new candidate for a hedging price is a risk-adjusted expectation.

- Let us define

$$U^{\pi,\nu}(t,s) := \mathbb{E}^{T-t,s}\left[\frac{V^{\pi}(T)Z^{\nu}(T)}{V^{\pi}(T-t)Z^{\nu}(T-t)}\right].$$

- $U^{\pi,\nu}$ is non-random.
- Not clear at this point how $U^{\pi,\nu}$ depends on market price of risk ν.
- We assume that $U^{\pi,\nu}$ satisfies the PDE

$$\begin{aligned}\frac{\partial U^{\pi,\nu}}{\partial t}(t,s) &= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} s_i s_j \alpha_{i,j}(T-t,s) D^2_{i,j} U^{\pi,\nu}(t,s) \\ &\quad + \sum_{i=1}^{n}\sum_{j=1}^{n} s_i \alpha_{i,j}(T-t,s)\pi_j(T-t,s) D_i U^{\pi,\nu}(t,s).\end{aligned}$$

Theorem: How does the optimal strategy look like?

For any strategy π there here exists a new strategy $\widehat{\pi}^{\nu}$ such that the corresponding wealth process $V^{\hat{v},\widehat{\pi}^{\nu}}$ with initial wealth $\hat{v} := U^{\pi,\nu}(T, S_0) \leq 1$ has the same value as V^{π} at time T, that is

$$V^{\hat{v},\widehat{\pi}^{\nu}}(T) = V^{\pi}(T).$$

$\widehat{\pi}^{\nu}$ takes the form

$$\widehat{\pi}_i^{\nu}(t,s) = s_i D_i \log U^{\pi,\nu}(T-t,s) + \pi_i(t,s)$$

and is optimal: There exists no strategy ρ such that

$$V^{\tilde{v},\rho}(T) \geq V^{\pi}(T) = V^{\hat{v},\widehat{\pi}^{\nu}}(T)$$

for some $\tilde{v} < \hat{v}$.

The proof relies on Itô's formula.

- Define the martingale $N^{\pi,\nu}$ as

$$N^{\pi,\nu}(t) := \mathbb{E}^s[V^\pi(T)Z^\nu(T)|\mathcal{F}(t)]$$

and compute its dynamics as

$$\frac{dN^{\pi,\nu}(t)}{N^{\pi,\nu}(t)} = \sum_{k=1}^{K}\left(\sum_{i=1}^{n} S_i(t)\sigma_{i,k}\frac{D_i U^{\pi,\nu}}{U^{\pi,\nu}} - \nu_k^\pi\right) dW_k(t) + C^{\pi,\nu}dt,$$

where $C^{\pi,\nu}(t,s)$ disappears because of the assumption that $U^{\pi,\nu}$ satisfies a PDE.

- But thus,

$$\frac{dN^{\pi,\nu}(t)}{N^{\pi,\nu}(t)} = -\sum_{k=1}^{K}\nu_k^{\widehat{\pi}^\nu}(t,S(t))dW_k(t),$$

which are the dynamics of $V^{\hat{\nu},\widehat{\pi}^\nu}(t)Z^\nu(t)$.

We can change the measure to compute $U^{\pi,\nu}$

- There exists not always an equivalent local martingale measure.
- However, after making some technical assumptions on the probability space and the filtration we can construct a new measure $\mathbb{Q}^{\nu}$ which corresponds to a "removal of the stock price drift".
- Based on the work of Föllmer and Meyer and along the lines of Delbaen and Schachermayer.

Theorem: Under a new measure $\mathbb{Q}^\nu$ the drifts disappear.

There exists a measure $\mathbb{Q}^\nu$ such that $\mathbb{P} \ll \mathbb{Q}^\nu$. More precisely, for all nonnegative $\mathcal{F}(T)$-measurable random variables Y we have

$$\mathbb{E}^{\mathbb{P}}[Z^\nu(T)Y] = \mathbb{E}^{\mathbb{Q}^\nu}\left[Y\mathbf{1}_{\left\{\frac{1}{Z^\nu(T)}>0\right\}}\right].$$

Under this measure $\mathbb{Q}^\nu$, the stock price processes follow

$$dS_i(t) = S_i(t)\sum_{k=1}^{K}\sigma_{i,k}(t,S(t))\,d\widetilde{W}_k(t)$$

up to time $\tau^\nu := \inf\{t \in [0;T] : 1/Z^\nu(t) = 0\}$. Here,

$$\widetilde{W}_k(t\wedge\tau^\nu) := W_k(t\wedge\tau^\nu) + \int_0^{t\wedge\tau^\nu}\nu_k(u,S(u))du$$

is a K-dimensional $\mathbb{Q}^\nu$-Brownian motion stopped at time τ^ν.

What happens in between time 0 and time T: Bayes' rule.

For all nonnegative $\mathcal{F}(T)$-measurable random variables Y the representation

$$\mathbb{E}^{\mathbb{Q}^\nu}\left[Y\mathbf{1}_{\{1/Z^\nu(T)>0\}} \middle| \mathcal{F}(t)\right] = \mathbb{E}^{\mathbb{P}}[Z^\nu(T)Y|\mathcal{F}(t)]\frac{1}{Z^\nu(t)}\mathbf{1}_{\{1/Z^\nu(t)>0\}}$$

holds $\mathbb{Q}^\nu$-almost surely (and thus $\mathbb{P}$-almost surely) for all $t \in [0; T]$.

The class of Bessel processes with drift provides interesting arbitrage opportunities.

- We begin with defining an auxiliary stochastic process X as

$$dX(t) = \left(\frac{1}{X(t)} - c \right) dt + dW(t)$$

  with W denoting a Brownian motion and $c \geq 0$ a constant.
- $X(t)$ is for all $t \geq 0$ strictly positive since X is a Bessel process under an equivalent measure.
- The stock price process is now defined via

$$dS(t) = \frac{1}{X(t)} dt + dW(t) = S(t) \left(\frac{1}{S^2(t) - S(t)ct} dt + \frac{1}{S(t)} dW(t) \right)$$

  with $S(0) = X(0) > 0$.

After a change of measure, the Bessel process becomes Brownian motion.

- As a reminder:

$$dS(t) = \frac{1}{S(t) - ct}dt + dW(t).$$

- We have $S(t) \geq X(t) > 0$ for all $t \geq 0$.
- The market price of risk is $\nu(t,s) = 1/(s - ct)$.
- Thus, the inverse stochastic discount factor $1/Z^{\nu}$ becomes zero exactly when $S(t)$ hits ct.
- Removing the drift with a change of measure as before makes S a Brownian motion (up to the first hitting time of zero by $1/Z^{\nu}$) under $\mathbb{Q}$.

The optimal strategy for getting one dollar at time T can be explicitly computed.

- For $\pi^1(t,s) \equiv 0$ we get

$$U^{\pi^1}(T-t,s) = \Phi\left(\frac{s-cT}{\sqrt{T-t}}\right) - \exp(2cs-2c^2t)\Phi\left(\frac{-s-cT+2c}{\sqrt{T-t}}\right.$$

- In the special case of $c=0$ this yields the optimal strategy

$$\widehat{\pi}^1(t,s) = \frac{2\frac{s}{\sqrt{T-t}}\phi\left(\frac{s}{\sqrt{T-t}}\right)}{2\Phi\left(\frac{s}{\sqrt{T-t}}\right)-1} > 0.$$

- This strategy can also be represented as

$$\widehat{\pi}^1(t,s) = 1 - s^2\mathbb{E}^{\mathbb{Q}^s}\left[\frac{1}{T_0}\middle| \min_{0\le u\le T-t} \widetilde{W}_u > 0\right]$$

where T_0 denotes the first hitting time of zero by $\widetilde{W}$.

Conclusion

- No equivalent local martingale measure needed to find an optimal hedging strategy based upon the familiar delta hedge.
- Allows for bigger class of models which include more realistic stock price models.
- The dynamics of stochastic processes under a non-equivalent measure and a generalized Bayes' rule might be of interest themselves.
- From an analytic point of view we have obtained results concerning non-uniqueness of a Cauchy problem.
- We have computed some optimal trading strategies in standard examples for which so far only ad-hoc and not necessarily optimal strategies have been known.

Thank you!