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Optimal trading strategies under arbitrage

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How should an investor trade and how much capital does she need?

- Imagine an investor who wants to hold the stock S_i with price $S_i(0)$ of a company in a year.
- Surely, she could just buy the stock today for a price $S_i(0)$.
- This might not be an "optimal strategy", even under a classical no-arbitrage situation ("no free lunch with vanishing risk").
- There can be other "strategies" which require less initial capital than $S_i(0)$ but enable her to hold the stock after one year.
- But how much initial capital does she need at least and how should she trade?

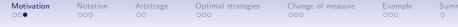
Motivation Notation Arbitrage Optimal strategies Change of measure Example Summary 000 Two examples where the stock price does not reflect the hedging price.

• Reciprocal of the three-dimensional Bessel process (NFLVR):

$$d\tilde{S}(t) = -\tilde{S}^2(t)dW(t)$$

• Three-dimensional Bessel process:

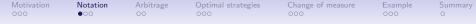
$$dS(t) = \frac{1}{S(t)}dt + dW(t)$$



How can a portfolio optimally been hedged?

- Optimality in the sense of smallest initial capital.
- Solved by Fernholz D. & Karatzas I. for the market portfolio.
- "Hedging price" characterized via a PDE which can allow for non-unique solutions.

- Change of measure to simplify computations.
- In Markovian framework.
- No martingale representation theorem used.



We assume a Markovian market model.

- Our time is finite: $T < \infty$. Interest rates are zero.
- The stocks $S(\cdot) = (S_1(\cdot), \dots, S_n(\cdot))^{\mathcal{T}}$ follow

$$dS_i(t) = S_i(t) \left(\beta_i(t, S(t))dt + \sum_{k=1}^K \sigma_{i,k}(t, S(t))dW_k(t)\right)$$

with some measurability and integrability conditions.

- \rightarrow Markovian
- but not necessarily complete (K > n allowed).
- The covariance process is defined as

$$\alpha_{i,j}(t,S(t)) := \sum_{k=1}^{K} \sigma_{i,k}(t,S(t)) \sigma_{j,k}(t,S(t))$$

Two important guys: the market price of risk and its close relative, the stochastic discount factor.

• The market price of risk is the function ν satisfying

$$\mathbb{P}\left(\left\{\beta = \sigma\nu\forall t \in [0; T]\right\} \bigcap \left\{\int_0^T \|\nu\|^2 dt < \infty\right\}\right) = 1.$$

- The market price of risk is not necessarily unique.
- Related is the stochastic discount factor

$$Z^{\nu}(t) := \exp\left(-\int_{0}^{t} \nu^{\mathcal{T}}(u, S(u)) dW(u) - \frac{1}{2} \int_{0}^{t} \|\nu(u, S(u))\|^{2} du\right)$$

with dynamics

Notation

$$dZ^{\nu}(t) = -\nu^{\mathcal{T}}(t, S(t))Z^{\nu}(t)dW(t)$$

• If Z^{ν} is a strict local martingale, arbitrage is possible.

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Everything an investor cares about: how and how much?

- We focus on Markovian trading strategies $\pi(t, S(t))$.
- Which percentage of the total wealth is invested in the single stocks depends on time and current market situation.
- The corresponding wealth process $V^{\nu,\pi}$ follows

$$\frac{dV^{\nu,\pi}(t)}{V^{\nu,\pi}(t)} = \sum_{i=1}^{n} \pi_i(t, S(t)) \frac{dS_i(t)}{S_i(t)}$$

and is usually not Markovian.

• We are ready to define the π -specific price of risk as

$$u^{\pi}(t,s) :=
u(t,s) - \sigma^{\mathcal{T}}(t,s)\pi(t,s).$$

Then

$$\frac{d(V^{\nu,\pi}(t)Z^{\nu}(t))}{V^{\nu,\pi}(t)Z^{\nu}(t)} = -\sum_{k=1}^{K} \nu_{k}^{\pi}(t,S(t))dW_{k}(t).$$



It can be shown that excluding arbitrage a priori leads to market models which cannot capture important properties of the real markets very well. (diversity, behavior of market weights)

- We call a strategy ρ with $\mathbb{P}(V^{\rho}(T) \ge V^{\pi}(T)) = 1$ and $\mathbb{P}(V^{\rho}(T) > V^{\pi}(T)) > 0$ relative arbitrage opportunity with respect to the strategy π .
- We call ρ a classical arbitrage opportunity if π invests fully in the bond, that is, if π(t, s) ≡ 0 for all (t, s) ∈ [0; T] × ℝⁿ₊.



But we exclude some forms of arbitrage.

- Remember: We have assumed that there exists some ν which maps the volatility into the drift, that is σ(·, ·)ν(·, ·) = β(·, ·).
- It can be shown that this assumption excludes "unbounded profit with bounded risk".
- Thus "making (a considerable) something out of almost nothing" is not possible.
- However, it is still possible to "certainly make something more out of something".

A new candidate for a hedging price is a risk-adjusted expectation.

• Let us define

$$U^{\pi,
u}(t,s):=\mathbb{E}^{T-t,s}\left[rac{V^{\pi}(T)Z^{
u}(T)}{V^{\pi}(T-t)Z^{
u}(T-t)}
ight].$$

- $U^{\pi,\nu}$ is non-random.
- Not clear at this point how U^{π,ν} depends on market price of risk ν.
- We assume that $U^{\pi,\nu}$ satisfies the PDE

$$\frac{\partial U^{\pi,\nu}}{\partial t}(t,s) = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} s_i s_j \alpha_{i,j} (T-t,s) D_{i,j}^2 U^{\pi,\nu}(t,s) + \sum_{i=1}^{n} \sum_{j=1}^{n} s_i \alpha_{i,j} (T-t,s) \pi_j (T-t,s) D_i U^{\pi,\nu}(t,s).$$



Theorem: How does the optimal strategy look like?

For any strategy π there here exists a new strategy $\hat{\pi}^{\nu}$ such that the corresponding wealth process $V^{\hat{v},\hat{\pi}^{\nu}}$ with initial wealth $\hat{v} := U^{\pi,\nu}(T,S_0) \leq 1$ has the same value as V^{π} at time T, that is

$$V^{\hat{\mathbf{v}},\widehat{\pi}^{\nu}}(T)=V^{\pi}(T).$$

 $\widehat{\pi}^{\nu}$ takes the form

$$\widehat{\pi}_i^
u(t,s) = s_i D_i \log U^{\pi,
u}(T-t,s) + \pi_i(t,s)$$

and is optimal: There exists no strategy ρ such that

$$V^{ ilde{
u},
ho}(T)\geq V^{\pi}(T)=V^{\hat{
u},\hat{\pi}^{
u}}(T)$$

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for some $\tilde{v} < \hat{v}$.

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The proof relies on Itô's formula.

• Define the martingale $N^{\pi,
u}$ as

$$N^{\pi,
u}(t) := \mathbb{E}^{s}[V^{\pi}(T)Z^{
u}(T)|\mathcal{F}(t)]$$

and compute its dynamics as

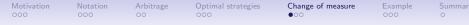
$$\frac{dN^{\pi,\nu}(t)}{N^{\pi,\nu}(t)} = \sum_{k=1}^{K} \left(\sum_{i=1}^{n} S_i(t) \sigma_{i,k} \frac{D_i U^{\pi,\nu}}{U^{\pi,\nu}} - \nu_k^{\pi} \right) dW_k(t) + C^{\pi,\nu} dt,$$

where $C^{\pi,\nu}(t,s)$ disappears because of the assumption that $U^{\pi,\nu}$ satisfies a PDE.

But thus,

$$rac{dN^{\pi,
u}(t)}{N^{\pi,
u}(t)}=-\sum_{k=1}^{K}
u_k^{\widehat{\pi}^
u}(t,S(t))dW_k(t),$$

which are the dynamics of $V^{\hat{\nu},\hat{\pi}^{\nu}}(t)Z^{\nu}(t)$.



We can change the measure to compute $U^{\pi, u}$

- There exists not always an equivalent local martingale measure.
- However, after making some technical assumptions on the probability space and the filtration we can construct a new measure \mathbb{Q}^{ν} which corresponds to a "removal of the stock price drift".
- Based on the work of Föllmer and Meyer and along the lines of Delbaen and Schachermayer.

Theorem: Under a new measure \mathbb{Q}^{ν} the drifts disappear.

There exists a measure \mathbb{Q}^{ν} such that $\mathbb{P} \ll \mathbb{Q}^{\nu}$. More precisely, for all nonnegative $\mathcal{F}(\mathcal{T})$ -measurable random variables Y we have

$$\mathbb{E}^{\mathbb{P}}[Z^{\nu}(T)Y] = \mathbb{E}^{\mathbb{Q}^{\nu}}\left[Y\mathbf{1}_{\left\{\frac{1}{Z^{\nu}(T)}>0\right\}}\right].$$

Under this measure $\mathbb{Q}^{
u}$, the stock price processes follow

$$dS_i(t) = S_i(t) \sum_{k=1}^{K} \sigma_{i,k}(t, S(t)) d\widetilde{W}_k(t)$$

up to time $au^
u := \inf\{t \in [0; T] : 1/Z^
u(t) = 0\}$. Here,

$$\widetilde{W}_k(t\wedge au^
u):=W_k(t\wedge au^
u)+\int_0^{t\wedge au^
u}
u_k(u,S(u))du$$

is a K-dimensional \mathbb{Q}^{ν} -Brownian motion stopped at time τ^{ν} .

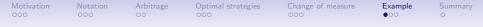


What happens in between time 0 and time T: Bayes' rule.

For all nonnegative $\mathcal{F}(T)$ -measurable random variables Y the representation

$$\mathbb{E}^{\mathbb{Q}^{\nu}}\left[\left.Y\mathbf{1}_{\left\{1/Z^{\nu}(\mathcal{T})>0\right\}}\right|\mathcal{F}(t)\right]=\mathbb{E}^{\mathbb{P}}[Z^{\nu}(\mathcal{T})Y|\mathcal{F}(t)]\frac{1}{Z^{\nu}(t)}\mathbf{1}_{\left\{1/Z^{\nu}(t)>0\right\}}$$

holds \mathbb{Q}^{ν} -almost surely (and thus \mathbb{P} -almost surely) for all $t \in [0; T]$.



The class of Bessel processes with drift provides interesting arbitrage opportunities.

• We begin with defining an auxiliary stochastic process X as

$$dX(t) = \left(\frac{1}{X(t)} - c\right)dt + dW(t)$$

with W denoting a Brownian motion and $c \ge 0$ a constant.

- X(t) is for all t ≥ 0 strictly positive since X is a Bessel process under an equivalent measure.
- The stock price process is now defined via

$$dS(t) = \frac{1}{X(t)}dt + dW(t) = S(t)\left(\frac{1}{S^2(t) - S(t)ct}dt + \frac{1}{S(t)}dW(t)\right)$$

with S(0) = X(0) > 0.

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After a change of measure, the Bessel process becomes Brownian motion.

• As a reminder:

$$dS(t) = \frac{1}{S(t) - ct} dt + dW(t).$$

- We have $S(t) \ge X(t) > 0$ for all $t \ge 0$.
- The market price of risk is $\nu(t,s) = 1/(s-ct)$.
- Thus, the inverse stochastic discount factor $1/Z^{\nu}$ becomes zero exactly when S(t) hits ct.
- Removing the drift with a change of measure as before makes S a Brownian motion (up to the first hitting time of zero by $1/Z^{\nu}$) under \mathbb{Q} .

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The optimal strategy for getting one dollar at time T can be explicitly computed.

• For $\pi^1(t,s) \equiv 0$ we get

$$U^{\pi^{1}}(T-t,s) = \Phi\left(\frac{s-cT}{\sqrt{T-t}}\right) - \exp(2cs - 2c^{2}t)\Phi\left(\frac{-s-cT+2c}{\sqrt{T-t}}\right)$$

• In the special case of c = 0 this yields the optimal strategy

$$\widehat{\pi}^1(t,s) = rac{2rac{s}{\sqrt{T-t}}\phi\left(rac{s}{\sqrt{T-t}}
ight)}{2\Phi\left(rac{s}{\sqrt{T-t}}
ight) - 1} > 0.$$

• This strategy can also be represented as

$$\widehat{\pi}^1(t,s) = 1 - s^2 \mathbb{E}^{\mathbb{Q}^s} \left[\left. \frac{1}{T_0} \right| \min_{0 \le u \le T-t} \widetilde{W}_u > 0
ight]$$

э.

where T_0 denotes the first hitting time of zero by W.



- No equivalent local martingale measure needed to find an optimal hedging strategy based upon the familiar delta hedge.
- Allows for bigger class of models which include more realistic stock price models.
- The dynamics of stochastic processes under a non-equivalent measure and a generalized Bayes' rule might be of interest themselves.
- From an analytic point of view we have obtained results concerning non-uniqueness of a Cauchy problem.
- We have computed some optimal trading strategies in standard examples for which so far only ad-hoc and not necessarily optimal strategies have been known.

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Thank you!

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