Hedging Options In The Incomplete Market With Stochastic Volatility

Rituparna Sen

Sunday, Nov 15

1. Motivation

- This is a pure jump model and hence avoids the theoretical drawbacks of continuous path models. For example, the quadratic variation is not observable.
- Take into account the fact that stock prices move on the discrete grid in multiples of tick.
- Unlike general models with jumps, one can set up derivative security hedging with Birth-Death process.

2. Birth & Death Model

Perrakis(1988), Korn et. al.(1998)
Stock Price S_t. Jump size $\pm c$.
$N_t = c \times S_t$ is a birth and death process.
Probability that jump size $Y_t = 1$ is p_t.
Such a process can be considered as a discretized version of the Black-Scholes model if the intensity of jumps is proportional to N_t^2.
Consider processes with intensity $\lambda_t N_t^2$.

- λ_t is a constant.

- λ_t a stochastic process and N_t is a birth and death process conditional on the λ_t process.

Let the measure associated with the process N_t be $\mathcal{P}$.
Let $\xi(t)$ be the underlying process of event times.
So $d\xi(t) = 1$ if there is a jump at time t.
$dS(t) = cY(t)d\xi(t)$
From martingale/ no arbitrage considerations, $p_t = \frac{1}{2}(1 + \frac{r_t - \frac{\lambda_t}{2}}{N_t \lambda_t})$ where r_t is the risk-free interest rate.

3. Convergence

Let the jump size go to zero and the rate of jumps go to infinity. Then the birth and death process described converges to geometric Brownian motion.

Denote $S_t^{(n)}$ the birth and death process with step size $c = 1/n$.
Let $X_t^{(n)} = ln(S_t^{(n)})$ and $X_t^{*(n)} = X_t^{(n)} - \int_0^t E[ln(1 + \frac{Y_u}{N_u})]N_u^2 \lambda_u du - X_0^{(n)}$
Step(1): $\forall u > 0, E[ln(1 + \frac{Y_u}{N_u})]N_u^2 \lambda_u = (2p_u - 1)N_u \lambda_u + O(1/n) \xrightarrow{P} r_u - \frac{\lambda_u}{2}$
Step(2): $\forall t > 0, [X_t^{*(n)}, X_t^{*(n)}]_t \xrightarrow{P} \int_0^t \lambda_u du$
Step(3): $X_t^{*(n)}$ local martingale.
(2),(3) and Thm VIII.3.12 of Jacod and Shiryaev imply

$$X^{*(n)} \overset{d}{\Longrightarrow} BM(0, \int_0^t \lambda_u du)$$

This and (1) imply $X^{(n)} \overset{d}{\Longrightarrow} X$, where $X_t = BM(X_0 + \int_0^t (r_u - \frac{\lambda_u}{2})du, \int_0^t \lambda_u du)$
$X^{(n)} \overset{d}{\Longrightarrow} X, S^{(n)} = exp(X^{(n)}) \overset{d}{\Longrightarrow} exp(X)$, since exp is continuous function.
By Ito's formula, $d(e^{X_t}) = S_t[(r_t - \frac{\lambda_t}{2})dt + \sqrt{\lambda_t}dWt] + \frac{1}{2}S_t \lambda_t dt = S_t r_t dt + S_t \sqrt{\lambda_t} dWt$

4. Edgeworth expansion for Option Prices

Let us define $X_t^{(n)} = \ln(\frac{N_t^{(n)}}{n})$

$$X_t^{*(n)} = X_t^{(n)} - X_0^{(n)} - \int_0^t [p_{u,N_u} \log(1 + \frac{1}{N_u}) + (1 - p_{u,N_u}) \log(1 - \frac{1}{N_u})] N_u^2 \sigma_u^2 du$$

where $p_{t,N_t} = \frac{1}{2}(1 + \frac{\rho_t}{N_t \sigma_t^2})$.

Let $\mathcal{C}$ be the class of functions g that satisfy the following:

(i) $\int \mid \hat{g}(x) \mid dx < \infty$, uniformly in $\mathcal{C}$, and $\{\sum_u x_u^2 \hat{g}(x), g \in \mathcal{C}\}$ is uniformly integrable (here, $\hat{g}$ is the Fourier transform of g, which must exist for each $g \in \mathcal{C}$); or

(ii) g nd g'' bounded, uniformy in $\mathcal{C}$, and with g'' equicontinuous almost everywhere (under Lebesgue measure).

Under assumptions (I1) and (I2), for any $g \in \mathcal{C}$,

$$\mathrm{E} g(X_T^{*(n)}) = \mathrm{E} g(N(0, \lambda T)) + o(1/n)$$

(I1) There are $\underline{k}, \bar{k}$, $\underline{k} < \lambda T < \bar{k}$ so that

$n \left(l_T^{(n)} - \lambda T \right) \mathbf{I}(\underline{k} \leq l_T^{(n)} \leq \bar{k})$ is uniformly integrable, where $l_T^{(n)} = (X^{*(n)}, X^{*(n)})_T$

(I2) For the same $\underline{k}, \bar{k}$,

$$\mathrm{P}(\underline{k} \leq (X^{*(n)}, X^{*(n)})_T \leq \bar{k}) = 1 - o(1/n)$$

5. Hedging

The market is complete when we add a market traded derivative security.
We can hedge an option by trading the stock, the bond and another option.
Let $F_2(x,t), F_3(x,t)$ be the prices of two options at time t when price of stock is cx.
Let $F_1(x,t) = cx$ be the price of the stock and $F_0(x,t)B_0 \exp\{-\int_0^t \rho_s ds\}$ be price of the bond.
Assume F_i are continuous in both arguments.
We shall construct a self financing risk-less portfolio

$$V(t) = \sum_{i=0}^{3} \phi^{(i)}(t) F_i(x,t)$$

Let $u^{(i)}(t) = \frac{\phi^{(i)}(t)F_i(x,t)}{V(t)}$ be the proportion of wealth invested in asset i. $\sum u^{(i)} = 1$
Since V_t is self financing,

$$\begin{aligned}
\frac{dV(t)}{V(t)} &= \sum_{i=0}^{3} u^{(i)}(t) \frac{dF(x,t)}{F(x,t)} \\
&= u^{(0)}(t)\rho_t dt + u^{(1)}(t)\frac{1}{cx}(dN_{1t} - dN_{2t}) \\
&\quad + \sum_{i=2}^{3} u^{(i)}(t)(\alpha_{F_i}(x,t)dt + \beta_{F_i}(x,t)dN_{1t} + \gamma_{F_i}(x,t)dN_{2t})
\end{aligned}$$

V_t is risk-less $\Longrightarrow$
$u^{(1)}(t)\frac{1}{cx} + \sum_{i=2}^{3} u^{(i)}(t)\beta_{F_i}(x,t) = 0,$
$-u^{(1)}(t)\frac{1}{cx} + \sum_{i=2}^{3} u^{(i)}(t)\gamma_{F_i}(x,t) = 0$
No arbitrage $\Longrightarrow$ $u^{(0)}(t)\rho_t dt + \sum_{i=2}^{3} u^{(i)}(t)\alpha_{F_i}(x,t) = \rho_t$
The hedge ratios are:

$$u^{(2)} = [(1 - \frac{\alpha_{F_2}}{\rho} - x\beta_{F_2})(1 - \frac{\gamma_{F_2} + \beta_{F_2}}{\gamma_{F_3} + \beta_{F_3}})]^{-1}$$

$$u^{(3)} = [(1 - \frac{\alpha_{F_3}}{\rho} - x\beta_{F_3})(1 - \frac{\gamma_{F_3} + \beta_{F_3}}{\gamma_{F_2} + \beta_{F_2}})]^{-1}$$

$$u^{(0)} = -\frac{1}{\rho_t}(u^{(2)}\alpha_{F_2} + u^{(3)}\alpha_{F_3})$$

$$u^{(1)} = -x(u^{(2)}\beta_{F_2} + u^{(3)}\beta_{F_3})$$

6. Stochastic Intensity

Now we consider the case where the unobserved intensity λ_t is a stochastic process.
We first assume a two state Markov model for λ_t as in *Naik(1993)*
Later we describe how we can have similar results for other models on λ_t eg. *Hull and White(1987)*
Suppose there is an unobserved state process θ_t which takes 2 values , say 0 and 1.
The transition matrix is $\mathbf{Q}$.
When $\theta_t = i$, $\lambda_t = \lambda_i$.
Counting process associated with θ_t is ζ_t.

Let us denote by $\{\mathcal{G}_t\}$ the complete filtration $\sigma(S_u, \lambda_u, 0 \leq u \leq t)$ and by P the probability measure on $\{\mathcal{G}_t\}$ associated with the process (S_t, λ_t).

We get two different values of the expected price under the two values of $\theta(0)$.
The θ process is unobserved.
We cannot invert an option to get $\theta(0)$ because it takes two discrete values.
Need to introduce $\pi_i(t) = P(\theta_t = i \mid \mathcal{F}_t)$ where $\mathcal{F}_t = \sigma(S_u, 0 \leq u \leq t)$
As shown in Snyder(1973), under any $\hat{P} \in \mathcal{P}$ the π_{it} process evolves as:
$d\pi_{1t} = a(t)dt + b(t,1)dN_{1t} + b(t,2)dN_{2t}$ where $a(t)$ and $b(t,i)$ are $\mathcal{F}_t$ adapted processes.

7. Bayesian Framework

As shown in Yashin(1970) and Elliott et. al.(1995), the posterior of $\theta_j(t)$ is given by:

$$\begin{aligned}\pi_j(t) &= \pi_j(0) + \int_0^t \sum_i q_{ij}\pi_i(u)du \\ &+ \int_0^t \pi_j(u)(\bar{\lambda}(u) - \lambda_j)N_u^2 du \\ &+ \sum_{0<u<t} b_j(u)\end{aligned}$$

where $\bar{\lambda}(t) = \sum_i \pi_i(t)\lambda_i$ and $b_j(u) = \pi_j(u-)\left(\frac{\lambda_j p_{\lambda_j}(S_{u-}\to S_u)}{\sum_i \pi_i(u)\lambda_i p_{\lambda_i}(S_{u-}\to S_u)} - 1\right)$
Thus, $a_j(u) = \sum_i q_{ij}\pi_i(u) + \int_0^t \pi_j(u)(\bar{\lambda}(u) - \lambda_j)N_u^2$
Now we can hedge as in the constant intensity case with modified hedge ratios. For hedging with stochastic intensity, same results as in the fixed λ case holds with α, β, γ replaced by $\tilde{\alpha}$, $\tilde{\beta}$, $\tilde{\gamma}$

$$\tilde{\alpha} = \pi_0\frac{\partial F_0}{\partial t} + \pi_1\frac{\partial F_1}{\partial t}$$

$$\tilde{\beta} = \pi_0\beta_{F_0}F_0 + \pi_1\beta_{F_1}F_1$$

$$\tilde{\gamma} = \pi_0\gamma_{F_0}F_0 + \pi_1\gamma_{F_1}F_1$$

In this setting we need one option and the stock to hedge an option and do not need to invert at all time points as would be case if we did not use the posterior.

8. Description of data

- The data was obtained from the optionmetrics database on 3 stocks: Ford (Dec 2002), IBM (June 2002) and ABMD(Feb 2003). The stock data is transaction by transaction. The option data is daily best bid and ask prices for all options traded on that day.

- The data is filtered for after hour and international market trading. The data now is on tradings in NASDAQ regular hours.

- The tick size is 1/16 for Ford and 1/100 for IBM and ABMD.

- We shall use the data for the first day of the month as training sample and for the rest of the days as test sample. Estimating risk-neutral parameters by inverting option prices in training sample.

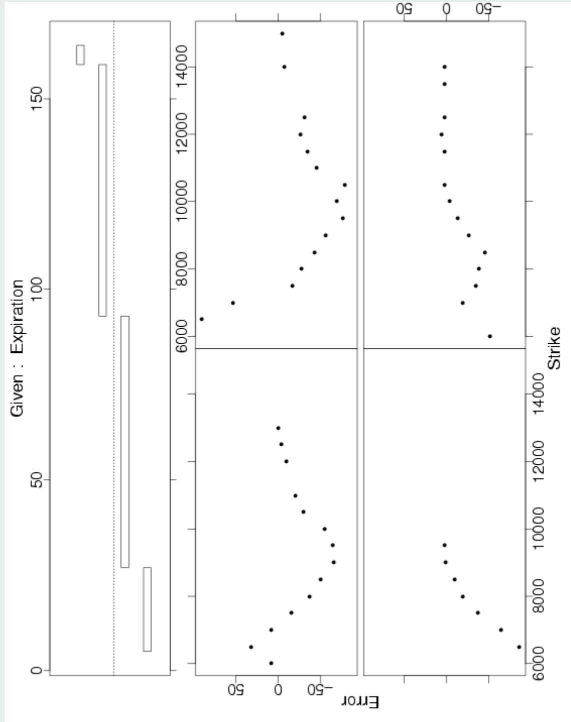

Figure 1: Error in CALL price for training sample of IBM data

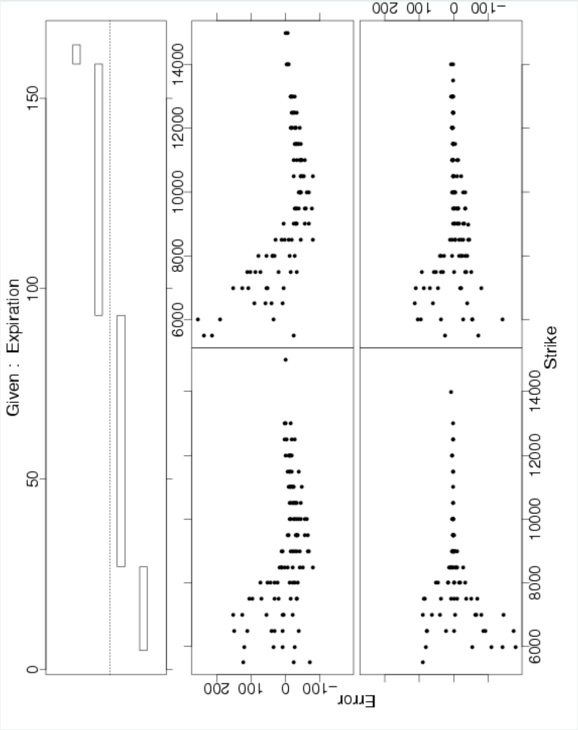

Figure 2: Error in CALL price for test sample of IBM data

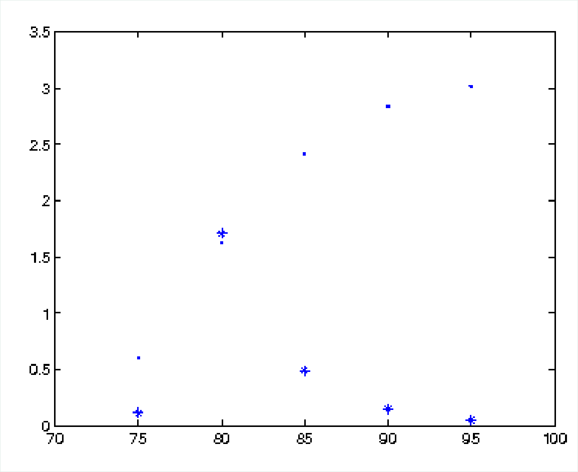

Figure 3: Hedging error of birth and death and the Black Scholes model, both with constant intensity rate.

9. Stochastic Intensity rate

- To estimate 5 parameters: λ_0, λ_1, q_{01}, q_{10}, and π.
- The objective is to find the parameter set that minimizes the root mean square error between the bid-ask-midpoint and the daily average of the predicted option price, for all options in the training sample.
- We followed a diagonally scaled steepest descent algorithm with central difference approximation to the differential.
- The starting values of λ_0, λ_1 are taken to be equal to the value of the estimator $\hat{\lambda}$ obtained in the constant intensity model.
- The starting values of q_{01}, q_{10} are obtained by a hidden Markov model approach using an iterative method (Ref Elliott 1995).
- We do a finite search on the parameter π.
- For the ABMD and Ford datasets, the RMSE of prediction obtained from the constant intensity method is less than the bid-ask spread.
- For IBM data, q_{01} , q_{10} and π are 8.64e-02, 1.2126 and λ_0 and λ_1 are 1.042039e-06 and 8.326884e-08.

- The RMSE is 29.6799. Compare this to the birth death model with constant intensity (RMSE=52.7729) or Black-Scholes model with constant volatility (RMSE=46.7688).

- This is an ill-posed problem.

10. Conclusions

- Both from Edgeworth expansions and real data examples, the pricing from this model is very similar to Black-Scholes pricing.
- However hedging is very different. Here we need an extra derivative to hedge an option.
- The introduction of Stochastic volatility does not necessitate the introduction of extra options for hedging purposes.
- Note that we are combining risk neutral estimation with updating by historical data