Regulation, Diversity and Arbitrage

Winslow Strong
Ph.D. Student at UC Santa Barbara
Advisor: Jean-Pierre Fouque

Third Western Conference in Mathematical Finance
Santa Barbara, California
November 14, 2009
Background

⇒ R. Fernholz [Fer99, Fer02]: diversity and equivalent martingale measures (EMMs) are incompatible.

⇒ His model: stocks are Itô processes ($\Rightarrow$ continuous), volatility is bounded, continuous trading, no transaction costs, no dividends, number of companies is constant.

⇒ Under these assumptions [FKK05] the only way diversity can be maintained is for the drifts to become unboundedly negative as stocks become large.

⇒ Motivating question: can diversity and no arbitrage coexist if diversity is maintained by a wealth-conserving redistribution of capital amongst companies?
Model Overview

➤ Start with a strongly Markovian stock model. Here we only consider solutions to SDEs.

➤ Regulation is imposed as a deterministic procedure occurring at the random time when relative capitalizations exit a permissible region.

➤ A regulatory event redistributes capital amongst companies. Total market value is conserved.

➤ The stock process forgets the past at regulation and its dynamics are completely determined by the post-regulation starting point.

➤ Portfolio cash flows are proportional to stock cash flows at a regulation event. Thus portfolio value is constant upon regulation even though stocks jump.

➤ This assumption is designed to mimic equity flows when a company is broken up into smaller parts (e.g. Bell Atlantic 1984).
The Unregulated (Pre)Model

Consider a market model \((\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0}, P, W, X)\) which is the unique strong solution to the SDE

\[
dX_{i,t} = X_{i,t} (b(X_t)dt + \sigma(X_t)dW_t), \quad X_0 = x, \quad 1 \leq i \leq n
\]

living in the positive orthant a.s. \(\forall t \geq 0\). \(W\) is a \(d\)-dimensional Brownian motion, \(d \geq n \geq 2\), and \(\mathbb{F}\) is the completed Brownian filtration.

- There is a money market account, \(B\), and furthermore we assume for simplicity that \(B_t = 1, \quad \forall t \geq 0\) corresponding to a risk-free rate of interest \(r \equiv 0\).

- We require that the volatility matrix, \(\sigma(x) \in \mathbb{R}^{n \times d}\), have full rank \((n)\), \(\forall x \in \Upsilon\).
Assume:

- Trading may occur in continuous time.
- Stocks pay no dividends
- There are no transaction costs.


$$M_t := \sum_{i=1}^{n} X_{i,t}, \quad \mu_{i,t} := \frac{X_{i,t}}{M_t}$$

$$X_t \in \Upsilon := \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1 > 0, \ldots, x_n > 0\} \quad \forall t \geq 0$$

$$\mu_t \in \Delta^n_+ := \left\{(\pi_1, \ldots, \pi_n) \in \mathbb{R}^n \mid \pi_1 > 0, \ldots, \pi_n > 0, \sum_{i}^{n} \pi_i = 1\right\} \quad \forall t \geq 0$$
Regulation Procedure

Confine market weights to $U^\mu$ by redistribution of capital amongst the stocks via a deterministic mapping, $\mathcal{R}^\mu$, upon exit from $U^\mu$. Total capital is conserved.

Definition 1. A regulation rule, $\mathcal{R}^\mu$, with respect to the open, nonempty set, $U^\mu \subset \Delta_+^n$, is a Borel function

$$\mathcal{R}^\mu : \Delta_+^n \setminus U^\mu \rightarrow U^\mu$$

This induces

$$U^x := \mu^{-1}(U^\mu) = \{x \in \mathcal{Y} \mid \mu(x) \in U^\mu\}$$

$$\mathcal{R}^x : \mathcal{Y} \setminus U^x \rightarrow U^x$$

$$\mathcal{R}^x(x) := \mathcal{R}^\mu(\mu(x)) \sum_{i=1}^{n} x_i$$
Either of \((U^\mu, R^\mu)\) or \((U^x, R^x)\) determine the same regulation rule, so we often refer to it as \((U, R)\).

- \(U^x\) is a conic region, i.e. \(x \in U^x \Rightarrow \lambda x \in U^x\), \(\forall \lambda > 0\), allowing any total market value, \(M\), for a given \(\mu \in U^\mu\).

- The regulation rule is first applied at the exit and stopping time

\[
\varsigma := \inf \{ t > 0 \mid \mu(X_t) \notin U^\mu \} = \inf \{ t > 0 \mid X_t \notin U^x \}
\]

- After \(\varsigma\) the regulated market model "resets" as if starting fresh from initial point \(R^x(X_\varsigma)\) until exit from \(U^x\) again.

- Applying this procedure inductively defines the law of the regulated stock price process on stochastic intervals via reference to the premodel law.
Regulated Market Model

\[ \tau_0 = 0, \quad W^1 := W, \quad X^1 = X, \quad \tau_1 := \varsigma_1 := \inf \{ t > 0 \mid X^1_t \notin U^x \} \]

By induction define the following \( k \geq 2 \), on \( \{\tau_{k-1} < \infty\} \),

\[ W^k_t := W_{\tau_{k-1}+t} - W_{\tau_{k-1}}, \quad \forall t \geq 0 \]

\[ dX^k_{i,t} = X^k_{i,t} (b(X^k_t)dt + \sigma(X^k_t)dW^k_t), \quad 1 \leq i \leq n, \quad X^k_0 = \mathfrak{R}^x (X^{k-1}_{\varsigma_{k-1}}) \]

\[ \varsigma_k := \inf \{ t > 0 \mid X^{k-1}_t \notin U^x \}, \quad \tau_k := \sum_{j=1}^{k} \varsigma_j \]

\( X^k \) is defined on \( \{\tau_{k-1} < \infty\} \) as the unique strong solution to the SDE above.
Explosions?

There is a possibility of explosion, i.e. of \( \lim_{k \to \infty} \tau_k < \infty \). To characterize this possibility define the following processes and variables

\[
N_t := \sum_{k=1}^{\infty} 1_{\{t > \tau_k\}} \in \mathcal{F}_t, \quad N_\infty := \lim_{t \to \infty} N_t
\]

The event \( \{N_\infty = k\} \) corresponds to exactly \( k \) exits occurring eventually in which case no further regulation is needed after the \( k \)th, and \( \tau_{k+1} = \infty \).

\[
\tau_\infty := \begin{cases} 
\infty & \text{on } \{N_\infty < \infty\} \\
\lim_{k \to \infty} \tau_k & \text{on } \{N_\infty = \infty\}
\end{cases}
\]
Regulated Stock Price Process

Definition 2. For regulation rule \((U, \mathcal{R})\) and initial point \(y_0 \in U^x\), the regulated stock price process is defined as

\[
Y_t(\omega) := X_0^1 \mathbf{1}_{\{0\}}(t) + \sum_{k=1}^{\infty} 1_{(\tau_{k-1}, \tau_k]}(\omega, t) X_{t-\tau_{k-1}}^k(\omega), \quad (\omega, t) \in [0, \tau_\infty).
\]

\[
Y_0 = X_0^1 = y_0
\]

\(a.s.\)

If \(P(\tau_\infty = \infty) = 1\) then we call the triple \((y_0, U, \mathcal{R})\) viable.
Portfolios in the Regulated Market

- Portfolio values are unaffected by a regulation event, mimicking a stock split.

- We want to recover the useful tool of representing the capital gains process as a stochastic integral.

- Define an effective stock process, \( \hat{Y} \), reflecting only the non-regulatory movements of \( Y \). Recalling that \( Y_{\tau_k^+} = \mathfrak{R}^x(Y^k_{\tau_k}) = X^{k+1}_0 \) on \( \{\tau_k < \infty\} \),

\[
\hat{Y}_t := Y_t - \sum_{k=1}^{N_t} (Y_{\tau_k^+} - Y_{\tau_k})
\]

\[= X^1_0 + \sum_{k=1}^{\infty} (X_{(0 \lor (t-\tau_{k-1}) \land s_k)}^k - X^k_0)\]
Definition 3. **Admissible trading strategies** in the regulated model are predictable processes $H$ such that

1. $H$ is $\hat{Y}$-integrable, that is, the stochastic integral
   \[ H \cdot \hat{Y} = (H \cdot \hat{Y})_{t \geq 0} := (\int_0^t H_s d\hat{Y}_s)_{t \geq 0} \] is well-defined in the sense of stochastic integration theory for semimartingales.

2. There is a constant, $K$, not depending on $t$ such that
   \[ (H \cdot \hat{Y})_t \geq -K, \quad \text{a.s., } \forall t \geq 0 \]

Definition 4. A **self-financing wealth processes** in the regulated model is any $V^H$ which satisfies:

\[ V_t^H = V_0^H + (H \cdot \hat{Y})_t \quad \forall t \geq 0. \]
EMMs in the Regulated Model

Assume that \((y_0, U, \mathcal{K})\) is viable, that is \(\tau_\infty = \infty\) a.s. We assumed that \(\sigma\) has full rank \((n)\) so there exists a market price of risk, \(\theta = \sigma'_t(\sigma_t \sigma'_t)^{-1}b_t\). When

\[
\int_0^T |\theta(Y_t)|^2 dt < \infty \quad \text{a.s. } \forall T > 0
\]

then we may define the local martingale and supermartingale,

\[
Z_t := \mathcal{E}(-(\theta(Y) \cdot W))_t = \exp \left\{ - \left( \int_0^t \theta(Y_s) dW_s + \frac{1}{2} \int_0^t |\theta(Y_s)|^2 ds \right) \right\}
\]
Proposition 1. If $Z$ is a martingale then the measure $Q$ generated from 
\[
\frac{dQ}{dP} := Z_T \]
is a local martingale measure for $\hat{Y}$ on horizon $[0, T]$.

The usual tools, e.g. the Kazamaki and Novikov criteria provide sufficient (although not necessary) conditions for $Z$ to be a martingale.
Diversity

**Definition 5.** A market model is diverse on horizon $T$ if there exists $\delta \in (0, 1)$ such that $\max_{1 \leq i \leq n}\{\mu_{i,t}\} < 1 - \delta$, $\forall t : 0 \leq t \leq T$ a.s. A market model is weakly diverse on horizon $T$ if there exists $\delta \in (0, 1)$ such that

$$\frac{1}{T} \int_0^T \max_{1 \leq i \leq n}\{\mu_{i,t}\} dt < 1 - \delta \ a.s.$$ 

- The regulatory procedure confines the market weights to $\bar{U}^\mu$, so it is easy to engineer diverse regulated markets.

- For example, fix any $\delta < \frac{n-1}{n}$ and let

$$U^\mu = \{\nu \in \Delta_+^n : \nu_i < 1 - \delta, 1 \leq i \leq n\}$$  \hspace{1cm} (4)
Regulated Volatility-Stabilized Markets

A non diverse market admitting relative arbitrage with respect to the market portfolio [FB08].

\[ dX_{i,t} = X_{i,t} \left[ \frac{1 + \alpha}{2 \mu_{i,t}} dt + \sqrt{\frac{1}{\mu_{i,t}}} dW_{i,t} \right], \quad 1 \leq i \leq n \]

for any constant \( \alpha \geq 0 \) with \( \mu_{i,t} = X_{i,t}/M_t, \) \( M_t = \sum_{i=1}^{n} X_{i,t}. \) This implies

\[ b_{i,t} = \frac{1 + \alpha}{2 \mu_{i,t}}, \quad \sigma_{i,\nu,t} = \delta_{i\nu} \sqrt{\frac{1}{\mu_{i,t}}}, \quad r_t = 0 \quad \text{and} \quad \theta_{\nu,t} = \frac{1 + \alpha}{2 \sqrt{\mu_{\nu,t}}} \quad 1 \leq \nu, i \leq n \]

This system has a weak solution that is unique in law [BP02].
Since we do not have strong existence or uniqueness, the construction presented herein can’t be used. Nevertheless by a more general construction, there exists \((\Omega, \mathcal{F}, \mathbb{F}, P, W, \{X^k\}_{1}^{\infty})\) satisfying the necessary properties to define regulated stock process \(Y\) as before.

Fixing some \(\varepsilon \in (0, \frac{1}{n})\), and some \(0 < \delta < (1 - \varepsilon) \wedge \frac{n-1}{n}\) we choose

\[
U^\mu := \{ \mu : \varepsilon < \mu_i < 1 - \delta, \sum_{i=1}^{n} \mu_i = 1 \} \subset \Delta^n_+
\]

\[
\mathcal{R}^\mu(\mu) = \mu_0 := \mu(y_0), \quad \forall \mu \in \Delta^n_+ \setminus U^\mu
\]

This simple regulation rule is viable. The rule above implies that \(\mu_t \in \bar{U}^\mu, \forall t \geq 0\), a.s. and so the regulated stock price process, \(Y\), is diverse.

\[
\int_{0}^{t} |\theta(Y_s)|^2 ds < \frac{Ct}{\varepsilon}, \quad \forall t \geq 0, \text{ a.s.}
\]
This implies that the Novikov condition

\[
E \left[ \exp \left\{ \frac{1}{2} \int_0^T |\theta(Y_s)|^2 ds \right\} \right] < \infty
\]

is satisfied here, and so the exponential local martingale \( Z \) is a martingale and generates an ELMM on \( \mathcal{F}_T \) by \( \frac{dQ}{dP} := Z_T \).

In fact, \( Q \) is a martingale measures here rather than merely a local martingale measure because

\[
\mu_{i,t} > \varepsilon \Rightarrow \sigma_{i,\nu,t} = \delta_{i\nu} \sqrt{\frac{1}{\mu_{i,t}}} < \frac{1}{\sqrt{\varepsilon}}.
\]
Conclusions

- EMMs (with respect to $\hat{Y}$) and diversity (with respect to $Y$) are compatible in this regulatory model.

- The key condition here is that the unregulated model be arbitrage free up until exit from $U$. In general additional regularity is needed.

- Do the conclusions for this type of regulatory model carry over to a model where companies are forced to split, with total market capital conserved?
References


