Regulation, Diversity and Arbitrage

Winslow Strong

Ph.D. Student at UC Santa Barbara

Advisor: Jean-Pierre Fouque

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Background

- R. Fernholz [Fer99, Fer02]: diversity and equivalent martingale measures (EMMs) are incompatible.
- His model: stocks are Ito processes ($\Rightarrow$continuous), volatility is bounded, continuous trading, no transaction costs, no dividends, number of companies is constant.
- Under these assumptions [FKK05] the only way diversity can be maintained is for the drifts to become unboundedly negative as stocks become large.
- Motivating question: can diversity and no arbitrage coexist if diversity is maintained by a wealth-conserving redistribution of capital amongst companies?

Model Overview

- Start with a strongly Markovian stock model. Here we only consider solutions to SDEs.
- Regulation is imposed as a deterministic procedure occurring at the random time when relative capitalizations exit a permissible region.
- A regulatory event redistributes capital amongst companies. Total market value is conserved.
- The stock process forgets the past at regulation and its dynamics are completely determined by the post-regulation starting point.
- Portfolio cash flows are proportional to stock cash flows at a regulation event. Thus portfolio value is constant upon regulation even though stocks jump.
- This assumption is designed to mimic equity flows when a company is broken up into smaller parts (e.g. Bell Atlantic 1984).

The Unregulated (Pre)Model

Consider a market model $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}, P, W, X)$ which is the unique strong solution to the SDE

$$dX_{i,t} = X_{i,t}\left(b(X_t)dt + \sigma(X_t)dW_t\right), \qquad X_0 = x, \qquad 1 \leq i \leq n$$

living in the positive orthant a.s. $\forall t \geq 0$. W is a d-dimensional Brownian motion, $d \geq n \geq 2$, and $\mathbb{F}$ is the completed Brownian filtration.

- There is a money market account, B, and furthermore we assume for simplicity that $B_t = 1, \quad \forall t \geq 0$ corresponding to a risk-free rate of interest $r \equiv 0$.

- We require that the volatility matrix, $\sigma(x) \in \mathbb{R}^{n\times d}$, have full rank (n), $\forall x \in \Upsilon$.

➡ Assume:

- ➥ Trading may occur in continuous time.
- ➥ Stocks pay no dividends
- ➥ There are no transaction costs.

➡ Market capitalization process: M. Market weight process: μ.

$$M_t := \sum_{i=1}^{n} X_{i,t}, \qquad \mu_{i,t} := \frac{X_{i,t}}{M_t}$$

$$X_t \in \Upsilon := \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1 > 0, \ldots, x_n > 0\} \qquad \forall t \geq 0$$

$$\mu_t \in \Delta_+^n := \left\{(\pi_1, \ldots, \pi_n) \in \mathbb{R}^n \mid \pi_1 > 0, \ldots, \pi_n > 0, \quad \sum_i^n \pi_i = 1\right\} \quad \forall t \geq 0$$

Regulation Procedure

Confine market weights to U^μ by redistribution of capital amongst the stocks via a deterministic mapping, $\mathfrak{R}^\mu$, upon exit from U^μ. Total capital is conserved.

Definition 1. *A regulation rule, $\mathfrak{R}^\mu$, with respect to the open, nonempty set, $U^\mu \subset \Delta_+^n$, is a Borel function*

$$\mathfrak{R}^\mu : \Delta_+^n \setminus U^\mu \to U^\mu$$

This induces

$$U^x := \mu^{-1}(U^\mu) = \{x \in \Upsilon \mid \mu(x) \in U^\mu\}$$

$$\mathfrak{R}^x : \Upsilon \setminus U^x \to U^x$$

$$\mathfrak{R}^x(x) := \mathfrak{R}^\mu(\mu(x)) \sum_{i=1}^n x_i$$

- Either of $(U^\mu, \mathfrak{R}^\mu)$ or $(U^x, \mathfrak{R}^x)$ determine the same regulation rule, so we often refer to it as $(U, \mathfrak{R})$.

- U^x is a conic region, i.e. $x \in U^x \Rightarrow \lambda x \in U^x$, $\forall \lambda > 0$, allowing any total market value, M, for a given $\mu \in U^\mu$.

- The regulation rule is first applied at the exit and stopping time

$$\varsigma := \inf\{t > 0 \mid \mu(X_t) \notin U^\mu\} = \inf\{t > 0 \mid X_t \notin U^x\}$$

- After ς the regulated market model "resets" as if starting fresh from initial point $\mathfrak{R}^x(X_\varsigma)$ until exit from U^x again.

- Applying this procedure inductively defines the law of the regulated stock price process on stochastic intervals via reference to the premodel law.

Regulated Market Model

$$\tau_0 = 0, \quad W^1 := W, \quad X^1 = X, \quad \tau_1 := \varsigma_1 := \inf\left\{t > 0 \mid X_t^1 \notin U^x\right\}$$

By induction define the following $k \geq 2$, on $\{\tau_{k-1} < \infty\}$,

$$W_t^k := W_{\tau_{k-1}+t} - W_{\tau_{k-1}}, \qquad \forall t \geq 0$$

$$dX_{i,t}^k = X_{i,t}^k \left(b(X_t^k)dt + \sigma(X_t^k)dW_t^k\right), \qquad 1 \leq i \leq n, \quad X_0^k = \mathfrak{R}^x(X_{\varsigma_{k-1}}^{k-1})$$

$$\varsigma_k := \inf\left\{t > 0 \mid X_t^{k-1} \notin U^x\right\}, \qquad \tau_k := \sum_{j=1}^{k} \varsigma_j$$

X^k is defined on $\{\tau_{k-1} < \infty\}$ as the unique strong solution to the SDE above.

Explosions?

There is a possibility of explosion, i.e. of $\lim_{k\to\infty}\tau_k < \infty$. To characterize this possibility define the following processes and variables

$$N_t := \sum_{k=1}^{\infty} \mathbf{1}_{\{t>\tau_k\}} \in \mathcal{F}_t, \qquad\qquad N_\infty := \lim_{t\to\infty} N_t$$

The event $\{N_\infty = k\}$ corresponds to exactly k exits occurring eventually in which case no further regulation is needed after the kth, and $\tau_{k+1} = \infty$.

$$\tau_\infty := \begin{cases} \infty & \text{on } \{N_\infty < \infty\} \\ \lim_{k\to\infty}\tau_k & \text{on } \{N_\infty = \infty\} \end{cases}$$

Regulated Stock Price Process

Definition 2. *For regulation rule $(U, \mathfrak{R})$ and initial point $y_0 \in U^x$, the* regulated stock price process *is defined as*

$$Y_t(\omega) := X_0^1 \mathbf{1}_{\{0\}}(t) + \sum_{k=1}^{\infty} \mathbf{1}_{(\tau_{k-1}, \tau_k]}(\omega, t) X^k_{t-\tau_{k-1}}(\omega), \quad (\omega, t) \in [0, \tau_\infty). \tag{1}$$

$$Y_0 = X_0^1 = y_0 \qquad \text{a.s.}$$

If $P(\tau_\infty = \infty) = 1$ then we call the triple $(y_0, U, \mathfrak{R})$ viable.

Portfolios in the Regulated Market

- Portfolio values are unaffected by a regulation event, mimicking a stock split.
- We want to recover the useful tool of representing the capital gains process as a stochastic integral.
- Define an effective stock process, $\hat{Y}$, reflecting only the non-regulatory movements of Y. Recalling that $Y_{\tau_k^+} = \mathfrak{R}^x(Y^k_{\tau_k}) = X_0^{k+1}$ on $\{\tau_k < \infty\}$,

$$\hat{Y}_t := Y_t - \sum_{k=1}^{N_t} (Y_{\tau_k^+} - Y_{\tau_k}) \tag{2}$$

$$= X_0^1 + \sum_{k=1}^{\infty} (X^k_{(0 \vee (t-\tau_{k-1})) \wedge \varsigma_k} - X_0^k) \tag{3}$$

Definition 3. Admissible trading strategies *in the regulated model are predictable processes* H *such that*

1. H *is* $\hat{Y}$*-integrable, that is, the stochastic integral* $H \cdot \hat{Y} = (H \cdot \hat{Y})_{t \geq 0} := (\int_0^t H_s d\hat{Y}_s)_{t \geq 0}$ *is well-defined in the sense of stochastic integration theory for semimartingales.*

2. *There is a constant,* K*, not depending on* t *such that*

$$(H \cdot \hat{Y})_t \geq -K, \qquad \text{a.s., } \forall t \geq 0$$

Definition 4. *A* self-financing *wealth processes in the regulated model is any* V^H *which satisfies:*

$$V_t^H = V_0^H + (H \cdot \hat{Y})_t \qquad \forall t \geq 0.$$

EMMs in the Regulated Model

Assume that $(y_0, U, \mathfrak{R})$ is viable, that is $\tau_\infty = \infty$ a.s. We assumed that σ has full rank (n) so there exists a market price of risk, $\theta = \sigma_t'(\sigma_t\sigma_t')^{-1}b_t$. When

$$\int_0^T |\theta(Y_t)|^2 dt < \infty \qquad \text{a.s. } \forall T > 0$$

then we may define the local martingale and supermartingale,

$$Z_t := \mathcal{E}(-(\theta(Y) \cdot W))_t = \exp\left\{-\left(\int_0^t \theta(Y_s)dW_s + \frac{1}{2}\int_0^t |\theta(Y_s)|^2 ds\right)\right\}$$

Proposition 1. *If Z is a martingale then the measure Q generated from $\frac{dQ}{dP} := Z_T$ is a local martingale measure for $\hat{Y}$ on horizon $[0, T]$.*

➡ The usual tools, e.g. the Kazamaki and Novikov criteria provide sufficient (although not necessary) conditions for Z to be a martingale.

Diversity

Definition 5. *A market model is* diverse *on horizon T if there exists $\delta \in (0,1)$ such that $\max_{1\leq i\leq n}\{\mu_{i,t}\} < 1-\delta$, $\forall t : 0 \leq t \leq T$ a.s. A market model is* weakly diverse *on horizon T if there exists $\delta \in (0,1)$ such that*

$$\frac{1}{T}\int_0^T \max_{1\leq i\leq n}\{\mu_{i,t}\}dt < 1-\delta \text{ a.s.}$$

- The regulatory procedure confines the market weights to $\bar{U}^{\mu}$, so it is easy to engineer diverse regulated markets.

- For example, fix any $\delta < \frac{n-1}{n}$ and let

$$U^{\mu} = \{\nu \in \Delta_+^n : \nu_i < 1-\delta, 1 \leq i \leq n\} \tag{4}$$

Regulated Volatility-Stabilized Markets

A non diverse market admitting relative arbitrage with respect to the market portfolio [FB08].

$$dX_{i,t} = X_{i,t}\left[\frac{1+\alpha}{2\mu_{i,t}}dt + \sqrt{\frac{1}{\mu_{i,t}}}dW_{i,t}\right], \qquad 1 \leq i \leq n$$

for any constant $\alpha \geq 0$ with $\mu_{i,t} = X_{i,t}/M_t$, $M_t = \sum_{i=1}^{n} X_{i,t}$. This implies

$$b_{i,t} = \frac{1+\alpha}{2\mu_{i,t}}, \quad \sigma_{i,\nu,t} = \delta_{i\nu}\sqrt{\frac{1}{\mu_{i,t}}}, \quad r_t = 0 \quad \text{and} \quad \theta_{\nu,t} = \frac{1+\alpha}{2\sqrt{\mu_{\nu,t}}} \qquad 1 \leq \nu, i \leq n \tag{5}$$

This system has a weak solution that is unique in law [BP02].

Since we do not have strong existence or uniqueness, the construction presented herein can't be used. Nevertheless by a more general construction, there exists $(\Omega, \mathcal{F}, \mathbb{F}, P, W, \{X^k\}_1^\infty)$ satisfying the necessary properties to define regulated stock process Y as before.

Fixing some $\varepsilon \in (0, \frac{1}{n})$, and some $0 < \delta < (1-\varepsilon) \wedge \frac{n-1}{n}$ we choose

$$U^\mu := \{\mu : \varepsilon < \mu_i < 1 - \delta, \quad \sum_{i=1}^n \mu_i = 1\} \subset \Delta_+^n$$

$$\mathfrak{R}^\mu(\mu) = \mu_0 := \mu(y_0), \qquad \forall \mu \in \Delta_+^n \setminus U^\mu$$

This simple regulation rule is viable. The rule above implies that $\mu_t \in \bar{U}^\mu, \forall t \geq 0,$ a.s. and so the regulated stock price process, Y, is diverse.

$$\int_0^t |\theta(Y_s)|^2 ds < \frac{Ct}{\varepsilon}, \qquad \forall t \geq 0, \text{ a.s.}$$

This implies that the Novikov condition

$$E\left[\exp\left\{\frac{1}{2}\int_0^T |\theta(Y_s)|^2 ds\right\}\right] < \infty$$

is satisfied here, and so the exponential local martingale Z is a martingale and generates an ELMM on $\mathcal{F}_T$ by $\frac{dQ}{dP} := Z_T$.

In fact, Q is a martingale measures here rather than merely a local martingale measure because

$$\mu_{i,t} > \varepsilon \Rightarrow \sigma_{i,\nu,t} = \delta_{i\nu}\sqrt{\frac{1}{\mu_{i,t}}} < \frac{1}{\sqrt{\varepsilon}}.$$

Conclusions

- EMMs (with respect to $\hat{Y}$) and diversity (with respect to Y) are compatible in this regulatory model.

- The key condition here is that the unregulated model be arbitrage free up until exit from U. In general additional regularity is needed.

- Do the conclusions for this type of regulatory model carry over to a model where companies are forced to split, with total market capital conserved?

References

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