Option Pricing Under a Stressed-Beta Model

Adam Tashman

in collaboration with

Jean-Pierre Fouque

University of California, Santa Barbara

Department of Statistics and Applied Probability

Center for Research in Financial Mathematics and Statistics

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Capital Asset Pricing Model (CAPM)

Discrete-time approach

Excess return of asset $R_a - R_f$ is linear function of excess return of market R_M and Gaussian error term:

$$R_a - R_f \quad = \quad \beta(R_M - R_f) + \epsilon$$

Beta coefficient estimated by regressing asset returns on market returns.

Difficulties with CAPM

Some difficulties with this approach, including:

1) Relationship between asset returns, market returns not always linear

2) Estimation of β from history, but future may be quite different

Ultimate goal of this research is to deal with both of these issues

Extending CAPM: Dynamic Beta

Two main approaches:

1) Retain linearity, but beta changes over time; Ferson (1989), Ferson and Harvey (1991), Ferson and Harvey (1993), Ferson and Korajczyk (1995), Jagannathan and Wang (1996)

2) Nonlinear model, by way of state-switching mechanism; Fridman (1994), Akdeniz, L., Salih, A.A., and Caner (2003)

ASC introduces *threshold CAPM model*. Our approach is related.

Estimating Implied Beta

Different approach to estimating β: look to options market

- *Forward-Looking Betas*, 2006
  P Christoffersen, K Jacobs, and G Vainberg
  Discrete-Time Model

- *Calibration of Stock Betas from Skews of Implied Volatilities*, 2009
  J-P Fouque, E Kollman
  Continuous-Time Model, stochastic volatility environment

Example of Time-Dependent Beta

Stock	Industry	Beta (2005-2006)	Beta (2007-2008)
AA	Aluminum	1.75	2.23
GE	Conglomerate	0.30	1.00
JNJ	Pharmaceuticals	-0.30	0.62
JPM	Banking	0.54	0.72
WMT	Retail	0.21	0.29

Larger β means greater sensitivity of stock returns relative to market returns

Regime-Switching Model

We propose a model similar to CAPM, with a key difference:

When market falls below level c, slope increases by δ, where $\delta > 0$

Thus, beta is two-valued

This simple approach keeps the mathematics tractable

Dynamics Under Physical Measure $\mathbb{P}$

M_t value of market at time *t*
S_t value of asset at time *t*

$$\frac{dM_t}{M_t} = \mu dt + \sigma_m dW_t$$ Market Model; const vol, for now

$$\frac{dS_t}{S_t} = \beta(M_t)\frac{dM_t}{M_t} + \sigma dZ_t$$ Asset Model

$\beta(M_t) = \beta + \delta\,\mathbb{I}_{\{M_t < c\}}$

Brownian motions W_t, Z_t indep: $d\left\langle W, Z\right\rangle_t = 0$

Dynamics Under Physical Measure $\mathbb{P}$

Substituting market equation into asset equation:

$$\frac{dS_t}{S_t} = \beta(M_t)\mu dt + \beta(M_t)\sigma_m dW_t + \sigma dZ_t$$

Asset dynamics depend on market level, market volatility σ_m

This is a geometric Brownian motion with volatility $\sqrt{\beta^2(M_t)\sigma_m^2 + \sigma^2}$

Note this is a stochastic volatility model

Dynamics Under Physical Measure $\mathbb{P}$

Process preserves the definition of β:

$$
\begin{aligned}
\frac{Cov\left(\frac{dS_t}{S_t}, \frac{dM_t}{M_t}\right)}{Var\left(\frac{dM_t}{M_t}\right)} &= \frac{Cov\left(\beta(M_t)\frac{dM_t}{M_t} + \sigma dZ_t, \frac{dM_t}{M_t}\right)}{Var\left(\frac{dM_t}{M_t}\right)} \\
&= \frac{Cov\left(\beta(M_t)\frac{dM_t}{M_t}, \frac{dM_t}{M_t}\right)}{Var\left(\frac{dM_t}{M_t}\right)} \quad \text{Since BM's indep} \\
&= \beta(M_t)
\end{aligned}
$$

Dynamics Under Risk-Neutral Measure $\mathbb{P}^\star$

Market is complete (M and S both tradeable)

Thus, $\exists$ unique Equivalent Martingale Measure $\mathbb{P}^\star$ defined as

$$\frac{d\mathbb{P}^\star}{d\mathbb{P}} = exp\left\{-\int_t^T \theta^{(1)} dW_s - \int_t^T \theta^{(2)} dZ_s - \frac{1}{2}\int_t^T \left\{(\theta^{(1)})^2 + (\theta^{(2)})^2\right\} ds\right\}$$

with

$$\begin{aligned} \theta^{(1)} &= \frac{\mu - r}{\sigma_m} \\ \theta^{(2)} &= \frac{r(\beta(M_t) - 1)}{\sigma} \end{aligned}$$

Dynamics Under Risk-Neutral Measure $I\!P^{\star}$

$$
\begin{aligned}
\frac{dM_t}{M_t} &= r dt + \sigma_m dW_t^* \\
\frac{dS_t}{S_t} &= r dt + \beta(M_t)\sigma_m dW_t^* + \sigma dZ_t^*
\end{aligned}
$$

where

$$
\begin{aligned}
dW_t^* &= dW_t + \frac{\mu - r}{\sigma_m} dt \\
dZ_t^* &= dZ_t + \frac{r(\beta(M_t) - 1)}{\sigma} dt
\end{aligned}
$$

By Girsanov's Thm, W_t^*, Z_t^* are indep Brownian motions under $I\!P^{\star}$.

Option Pricing

P price of option with expiry T, payoff $h(S_T)$

Option price at time $t < T$ is function of t, M, and S

(M,S) Markovian

Option price discounted expected payoff under risk-neutral measure $\mathbb{P}^*$

$$P(t, M, S) = I\!E^{\star} \left\{ e^{-r(T-t)} h(S_T) | M_t = M, S_t = S \right\}$$

State Variables

Define new state variables: $X_t = \log S_t$, $\xi_t = \log M_t$
Initial conditions $X_0 = x$, $\xi_0 = \xi$

Dynamics are:

$$\begin{aligned} d\xi_t &= \left(r - \frac{\sigma_m^2}{2}\right) dt + \sigma_m dW_t^* \\ dX_t &= \left(r - \frac{1}{2}(\beta^2(e^{\xi_t})\sigma_m^2 + \sigma^2)\right) dt + \beta(e^{\xi_t})\sigma_m dW_t^* + \sigma dZ_t^* \end{aligned}$$

State Variables

WLOG, let $t = 0$

In integral form,

$$\xi_t = \xi + \left(r - \frac{\sigma_m^2}{2}\right) t + \sigma_m W_t^*$$

Next, consider X at expiry (integrate from 0 to T):

$$\begin{aligned} X_T &= x + \left(r - \frac{\sigma^2}{2}\right) T - \frac{\sigma_m^2}{2} \int_0^T \beta^2(e^{\xi_t}) dt \\ &+ \sigma_m \int_0^T \beta(e^{\xi_t}) dW_t^* + \sigma Z_T^* \end{aligned}$$

Working with X_T

$$M_t < c \quad \Rightarrow \quad e^{\xi_t} < c \quad \Rightarrow \quad \xi_t < \log c$$

$$\beta(M_t) = \beta + \delta\, \mathbb{I}_{\{M_t < c\}} \quad \Rightarrow \quad \beta(e^{\xi_t}) = \beta + \delta\, \mathbb{I}_{\{\xi_t < \log c\}}$$

Using this definition for $\beta(e^{\xi_t})$, X_T becomes

$$\begin{aligned} X_T &= x + \left(r - \frac{\beta^2\sigma_m^2 + \sigma^2}{2}\right) T + \sigma_m \beta W_T^* + \sigma Z_T^* \\ &- (\delta^2 + 2\delta\beta)\frac{\sigma_m^2}{2} \int_0^T \mathbb{I}_{\{\xi_t < \log c\}} dt + \sigma_m \delta \int_0^T \mathbb{I}_{\{\xi_t < \log c\}} dW_t^* \end{aligned}$$

Occupation Time of Brownian Motion

Expression for X_T involves integral $\int_0^T \mathbb{I}_{\{\xi_t < \log c\}} dt$

This is occupation time of Brownian motion with drift

To simplify calculation, apply Girsanov to remove drift from ξ

Occupation Time of Brownian Motion

Consider new probability measure $\widetilde{I\!P}$ defined as

$$\frac{d\widetilde{I\!P}}{dI\!P^{\star}} = exp\left\{-\theta W_T^* - \frac{1}{2}\theta^2 T\right\}$$

$$\theta = \frac{1}{\sigma_m}\left(r - \frac{\sigma_m^2}{2}\right)$$

Under this measure, ξ_t is a martingale with dynamics

$$d\xi_t = \sigma_m d\widetilde{W}_t$$

$$d\widetilde{W}_t = dW_t^* + \frac{1}{\sigma_m}\left(r - \frac{\sigma_m^2}{2}\right) dt$$

Changing Measure: $\mathbb{P}^\star \to \widetilde{\mathbb{P}}$

Since W^* and Z^* indep, Z^* not affected by change of measure

Can replace Z^* with $\widetilde{Z}$

Under $\widetilde{\mathbb{P}}$,

$$
\begin{aligned}
X_T &= x + A_1 T + \sigma_m \beta \widetilde{W}_T \\
&+ \sigma \widetilde{Z}_T - A_2 \int_0^T \mathbb{I}_{\{\xi_t < \log c\}} dt \\
&+ \sigma_m \delta \int_0^T \mathbb{I}_{\{\xi_t < \log c\}} d\widetilde{W}_t
\end{aligned}
$$

where constants A_1, A_2 defined as

$$
\begin{aligned}
A_1 &= r(1-\beta) - \frac{\sigma_m^2(\beta^2 - \beta) + \sigma^2}{2} \\
A_2 &= \delta(\delta + 2\beta - 1)\frac{\sigma_m^2}{2} + \delta r
\end{aligned}
$$

First Passage Time

Now that ξ_t is driftless, easier to work with occupation time

Run process until first time it hits level $log\, c$

Denote this first passage time

$$\tau = \inf \{t \geq 0 : \xi_t = \log c\} = \inf \left\{t \geq 0 : \widetilde{W}_t = \tilde{c}\right\}$$

where

$$\tilde{c} = \frac{\log c - \xi}{\sigma_m}$$

Density of first passage time of $\xi_t = \xi$ to level $\log\, c$ is

$$p(u; \tilde{c}) = \frac{|\tilde{c}|}{\sqrt{2\pi u^3}} \exp\left(-\frac{\tilde{c}^2}{2u}\right), \quad u > 0$$

Including First Passage Time Information

First passage time τ may happen after T, so need to be careful
Can partition time horizon into two pieces:

$$[0, \tau \wedge T] \quad \text{and} \quad [\tau \wedge T, T]$$

If $\xi_t < \log c$, $\quad \tau \wedge T$ counts as occupation time

Including First Passage Time Information

Incorporating this information into X_T yields

$$\begin{aligned} X_T &= x + A_1 T + \sigma_m \beta \widetilde{W}_T + \sigma \widetilde{Z}_T \\ &\quad - A_2 (\tau \wedge T)\, \mathbb{I}_{\{\tilde{c}>0\}} - A_2 \int_{\tau\wedge T}^{T} \mathbb{I}_{\left\{\widetilde{W}_t<\tilde{c}\right\}} dt \\ &\quad + \sigma_m \delta\, \widetilde{W}_{\tau\wedge T}\, \mathbb{I}_{\{\tilde{c}>0\}} + \sigma_m \delta \int_{\tau\wedge T}^{T} \mathbb{I}_{\left\{\widetilde{W}_t<\tilde{c}\right\}} d\widetilde{W}_t \end{aligned}$$

Working with the Stochastic Integral

Stochastic integral can be re-expressed in terms of local time $\widetilde{L}^{\tilde{c}}$ of $\widetilde{W}$ at level $\tilde{c}$.

Applying Tanaka's formula to $\phi(w) = (w - \tilde{c})\mathbb{I}_{\{w<\tilde{c}\}}$ between $\tau \wedge T$ and T, we get:

$$\int_{\tau\wedge T}^{T} \mathbb{I}_{\{\widetilde{W}_t<\tilde{c}\}} d\widetilde{W}_t = \phi(\widetilde{W}_T) - \phi(\widetilde{W}_{\tau\wedge T}) + \widetilde{L}_T^{\tilde{c}} - \widetilde{L}_{\tau\wedge T}^{\tilde{c}} \,.$$

Starting Level of Market: Three Cases

Consider separately the three cases $\xi = \log c$, $\xi > \log c$, and $\xi < \log c$ (or equivalently $\tilde{c} = 0$, $\tilde{c} < 0$, $\tilde{c} > 0$)

Notation for terminal log-stock price, given ξ

Case $\xi = \log c$	terminal log-stock price Ψ_0
Case $\xi > \log c$	terminal log-stock price Ψ^+
Case $\xi < \log c$	terminal log-stock price Ψ^-

Consider Case $\xi < \log c$ as Example

In this case, $\tilde{c} > 0$ and we have

$$
\begin{aligned}
X_T \quad = \quad & x + A_1 T + \sigma_m \beta \widetilde{W}_T + \sigma \widetilde{Z}_T \\
& - A_2 (\tau \wedge T) - A_2 \int_{\tau \wedge T}^{T} \mathbb{I}_{\{\widetilde{W}_t < \tilde{c}\}} dt + \sigma_m \delta \widetilde{W}_{\tau \wedge T} \\
& + \sigma_m \delta \left[\left(\widetilde{W}_T - \tilde{c} \right) \mathbb{I}_{\{\widetilde{W}_T < \tilde{c}\}} - \left(\widetilde{W}_{\tau \wedge T} - \tilde{c} \right) \mathbb{I}_{\{\widetilde{W}_{\tau \wedge T} < \tilde{c}\}} + \widetilde{L}_T^{\tilde{c}} - \widetilde{L}_{\tau \wedge T}^{\tilde{c}} \right]
\end{aligned}
$$

Treat separately cases $\{\tau < T\}$ and $\{\tau > T\}$

Case $\xi < \log c$, contd.

- On $\{\tau > T\}$, we have:

$$\begin{aligned} X_T &= x + (A_1 - A_2)T + \sigma_m(\beta + \delta)\,\widetilde{W}_T + \sigma \widetilde{Z}_T \\ &=: \Psi^-_{T^+}(\widetilde{W}_T, \widetilde{Z}_T), \end{aligned}$$

where lower index T^+ stands for $\tau > T$

Distribution of X_T is given by distn of independent Gaussian r.v. $\widetilde{Z}_T$, and conditional distn of $\widetilde{W}_T$ given $\{\tau > T\}$.

Case $\xi < \log c$, contd.

Conditional distn of $\widetilde{W}_T$ given $\{\tau > T\}$:

From Karatzas and Shreve, one easily obtains:

$$\begin{aligned} \mathbb{P}\left\{\widetilde{W}_T \in da, \tau > T\right\} &= \frac{1}{\sqrt{2\pi T}}\left(e^{-\frac{a^2}{2T}} - e^{-\frac{(2\tilde{c}-a)^2}{2T}}\right) da, \quad a < \tilde{c}, \\ &=: q_T(a;\tilde{c})\, da \end{aligned}$$

Case $\xi < \log c$, contd.

- On $\{\tau = u\}$ with $u \leq T$, we have $\widetilde{W}_u = \tilde{c}$, and

$$
\begin{aligned}
X_T &= x + (A_1 - A_2)T + \sigma_m(\beta + \delta)\tilde{c} + \sigma_m \beta(\widetilde{W}_T - \widetilde{W}_u) + \sigma \widetilde{Z}_T \\
&\quad + A_2 \int_u^T \mathbb{I}_{\left\{\widetilde{W}_t - \widetilde{W}_u > 0\right\}} dt \\
&\quad + \sigma_m \delta \left[\left(\widetilde{W}_T - \widetilde{W}_u \right) \mathbb{I}_{\{\widetilde{W}_T - \widetilde{W}_u < 0\}} + \widetilde{L}_T^{\tilde{c}} - \widetilde{L}_u^{\tilde{c}} \right]
\end{aligned}
$$

Distn of X_T given by distn of $\widetilde{Z}_T$ and indep triplet $\left(B_{T-u}, L^0_{T-u}, \Gamma^+_{T-u}\right)$

Triplet comprised of value, local time at 0, and occupation time of positive half-space, at time $T-u$, of standard Brownian motion B.

Case $\xi < \log c$, contd.

In distribution:

$$\begin{aligned} X_T &= x + (A_1 - A_2)T + \sigma_m(\beta+\delta)\tilde{c} + \sigma_m B_{T-u}\left(\beta + \delta\,\mathbb{I}_{\{B_{T-u}<0\}}\right) + \sigma \widetilde{Z}_T \\ &\quad + A_2\,\Gamma^{+}_{T-u} + \sigma_m \delta L^{0}_{T-u} \\ &=: \Psi^{-}_{T^{-}}(B_{T-u}, L^{0}_{T-u}, \Gamma^{+}_{T-u}, \widetilde{Z}_T). \end{aligned}$$

Distn of triplet $\left(B_{T-u}, L^{0}_{T-u}, \Gamma^{+}_{T-u}\right)$ developed in paper by Karatzas and Shreve.

Karatzas-Shreve Triplet (1984)

$$\mathbb{P}\left\{\widetilde{W}_T \in da,\ \widetilde{L}^0_T \in db,\ \widetilde{\Gamma}^+_T \in d\gamma\right\}$$

$$= \begin{cases} 2p(T-\gamma; b)\, p(\gamma; a+b) & \text{if} \quad a > 0, b > 0, 0 < \gamma < T, \\ 2p(\gamma; b)\, p(T-\gamma; -a+b) & \text{if} \quad a < 0, b > 0, 0 < \gamma < T, \end{cases}$$

where $p(u; \cdot)$ is first passage time density

Back to Option Pricing Formula

Given final expression for X_T, option price at time $t = 0$ is

$$
\begin{aligned}
P_0 &= \mathbb{E}^\star \left\{ e^{-rT} h(S_T) \right\} \\
&= \widetilde{\mathbb{E}} \left\{ e^{-rT} h(e^{X_T}) \frac{d\mathbb{P}^\star}{d\widetilde{\mathbb{P}}} \right\} \\
&= \widetilde{\mathbb{E}} \left\{ e^{-rT} h(e^{X_T}) e^{\theta \widetilde{W}_T - \frac{1}{2}\theta^2 T} \right\} \\
&= e^{-rT} e^{-\frac{1}{2}\theta^2 T} \widetilde{\mathbb{E}} \left\{ h(e^{X_T}) e^{\theta \widetilde{W}_T} \right\}
\end{aligned}
$$

Option Pricing Formula, contd.

Decompose expectation on $\{\tau \leq T\}$ and $\{\tau > T\}$,

Denote by $n_T(z)$ the $\mathcal{N}(0,T)$ density,

Define the following convolution relation involving the K-S triplet:

$$
\begin{aligned}
&\int_0^{T-\gamma} g(a,b,\gamma;T-u)p(u;\tilde{c})du \\
&= \begin{cases} 2p(\gamma;a+b)\,p(T-\gamma;b+|\tilde{c}|) & \text{if} \quad a>0 \\ 2p(\gamma;b)\,p(T-\gamma;-a+b+|\tilde{c}|) & \text{if} \quad a<0 \end{cases} \\
&=: G(a,b,\gamma;T)
\end{aligned}
$$

Option Pricing Formula, contd.

The option pricing formula becomes

$$P_0 = e^{-(r+\frac{1}{2}\theta^2)T} \left[e^{\theta\tilde{c}} \int_{-\infty}^{\infty} \int_0^T \int_0^{\infty} \int_{-\infty}^{\infty} h(e^{\Psi_{T^-}^{\pm}(a,b,\gamma,z)}) e^{\theta a} \right.$$

$$\times G(a, b, \gamma; T)\, da\, db\, d\gamma\, n_T(z) dz$$

$$\left. + \left(\int_{-\infty}^{\infty} \int_{D^\pm} h(e^{\Psi_{T^+}^{\pm}(a,z)}) e^{\theta a} q_T(a; \tilde{c}) da\, n_T(z) dz \right) \right]$$

where

$$D^\pm = \begin{cases} (-\infty, \tilde{c}) & \text{if} \quad \tilde{c} > 0 \\ (\tilde{c}, \infty) & \text{if} \quad \tilde{c} < 0 \end{cases}$$

Note About Market Stochastic Volatility (SV)

- Assumption of constant market volatility σ_m not realistic
- Let market volatility be driven by fast mean-reverting factor
- Introducing market SV in model has effect on asset price dynamics
- To leading order, these prices are given by risk-neutral dynamics with σ_m replaced by *adjusted effective volatility* σ^* (see Fouque, Kollman (2009) for details)
- One could derive a formula for first-order correction, but formula is quite complicated and numerically involved

Market Implied Volatilities

Following Fouque, Papanicolaou, Sircar (2000) and Fouque, Kollman (2009), introduce *Log-Moneyness to Maturity Ratio* ($LMMR$)

$$LMMR = \frac{\log(K/x)}{T}$$

and for calibration purposes, we use $affine\ LMMR\ formula$

$$I \sim b^* + a^\epsilon\, LMMR$$

with intercept b^* and slope a^ϵ to be fitted to skew of options data

Then estimate adjusted effective volatility as

$$\sigma^* \sim b^* + a^\epsilon \left(r - \frac{b^{*2}}{2} \right)$$

Numerical Results and Calibration

Asset Skews of Implied Volatilities

Using Stressed-Beta model, price European call option

Use following parameter settings:

c	S_0	r	β	σ_m	σ	T
1000	100	0.01	1.0	0.30	0.01	1.0

$K = 70, 71, \ldots, 150$ to build implied volatility curves

Figure 1: Implied Volatility Skew vs. $\delta \quad (M_0 = c)$

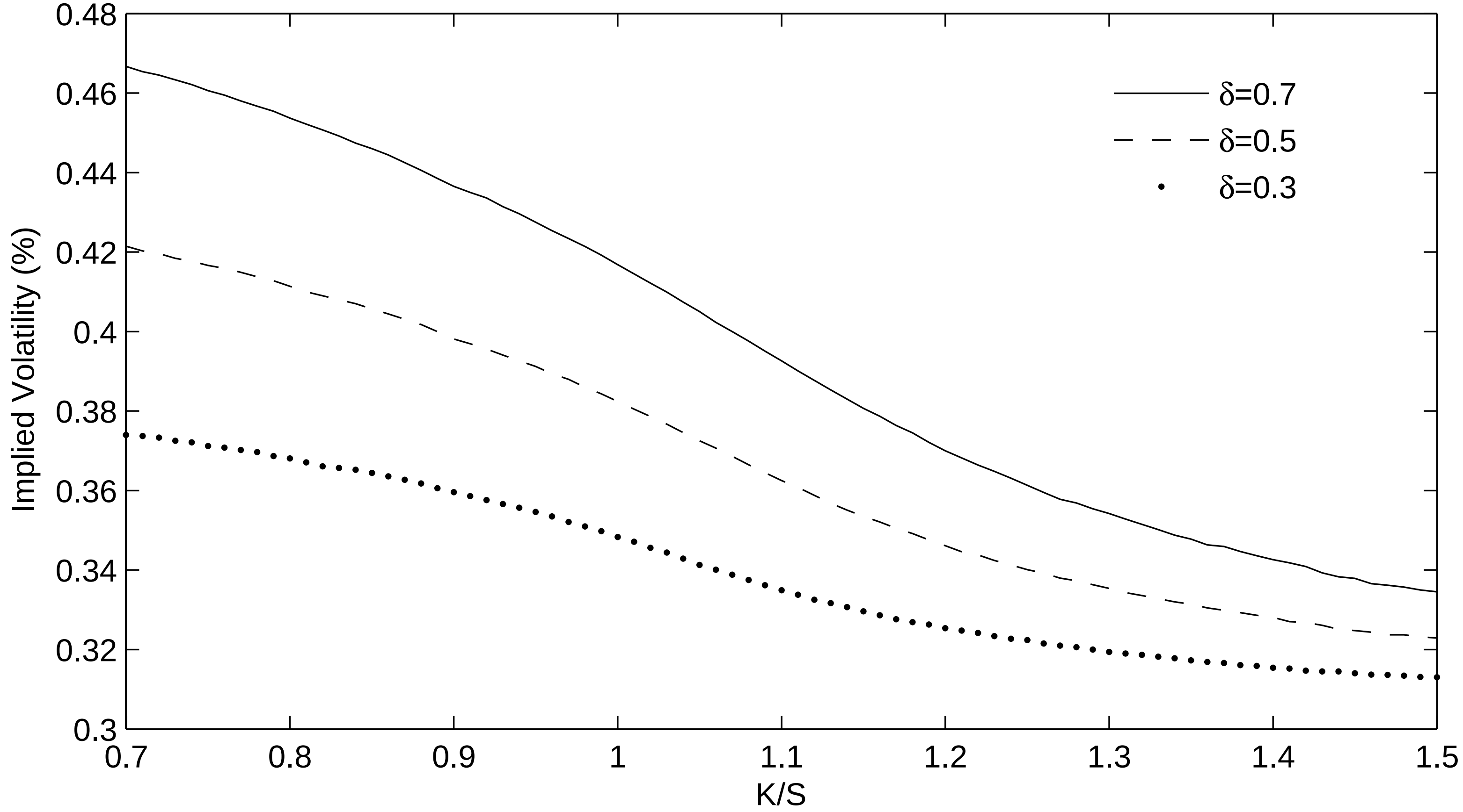
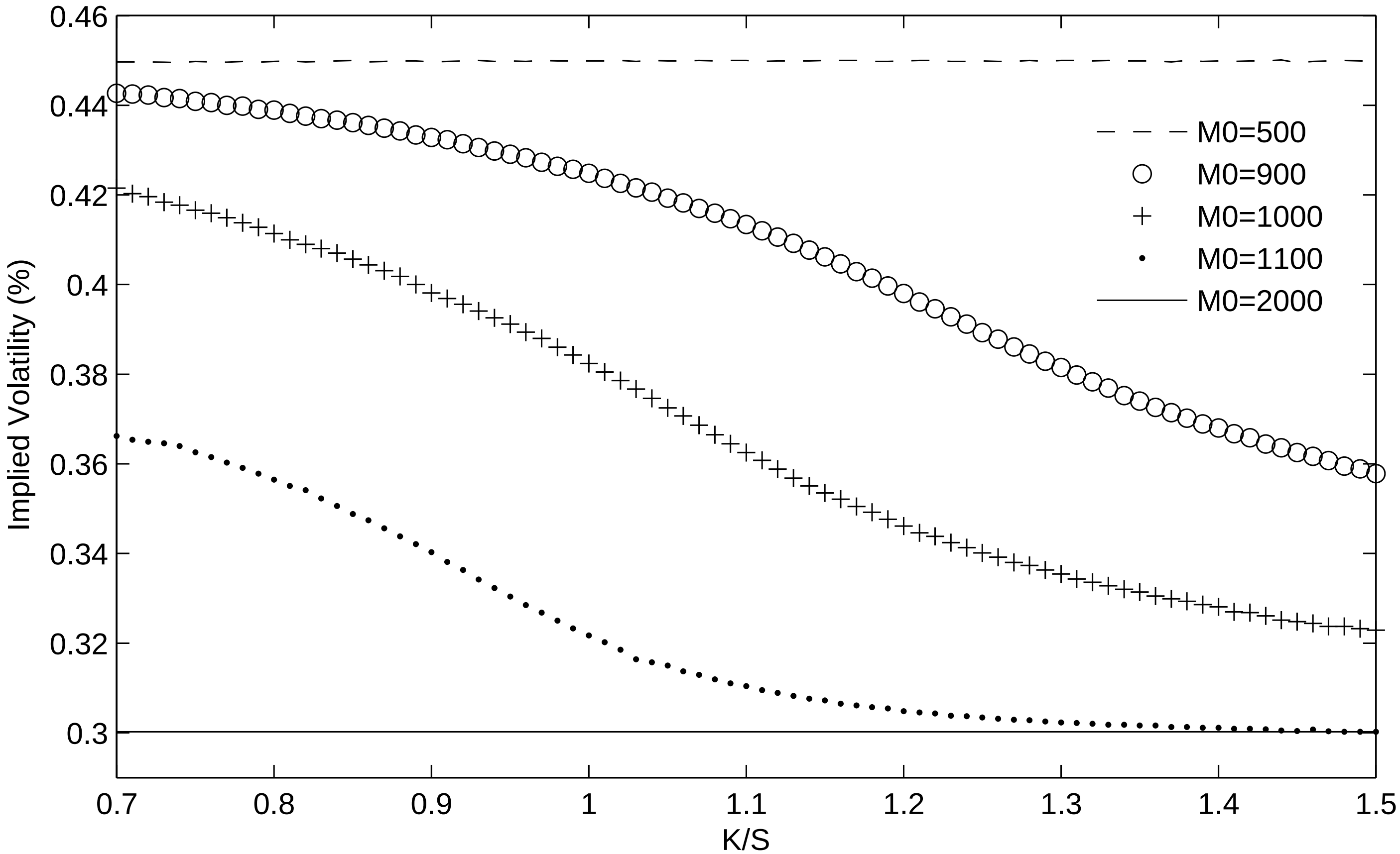


Figure 2: Implied Volatility Versus Starting Market $(\delta = 0.5)$

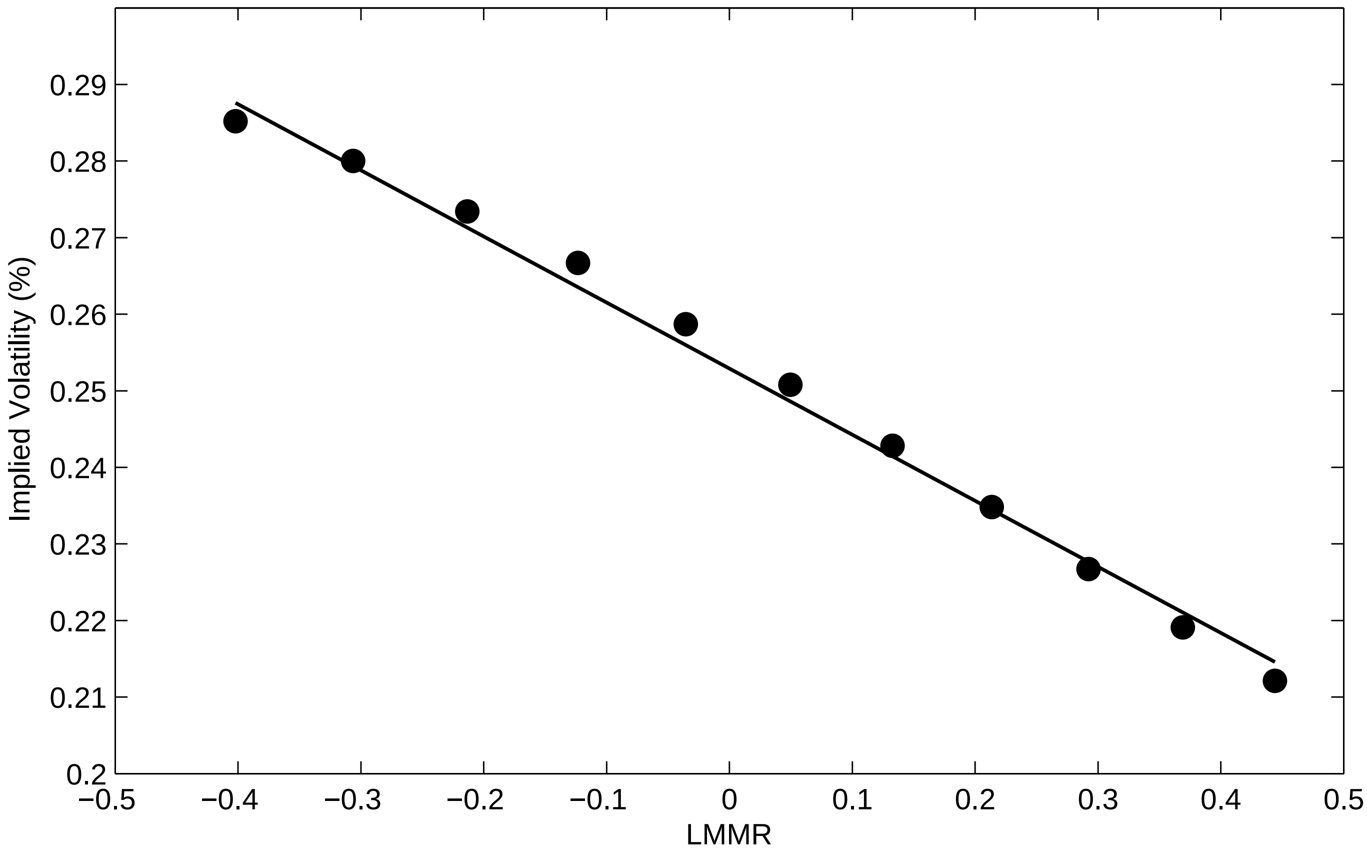


Calibration to Data: Amgen

- Consider Amgen call options with October 2009 expiry
- Strikes: Take options with $LMMR$ between -1 and 1, using closing mid-prices as of May 26, 2009
- For simplicity, asset-specific volatility $\sigma = 0$
- Market volatility σ^* estimated from call option data on S&P 500 Index (closest expiry Sep09)

From affine LMMR, $\sigma^* = 0.2549$

Figure 3: Affine $LMMR$ Fit to S&P 500 Index Options



Calibration to Data: Amgen, contd.

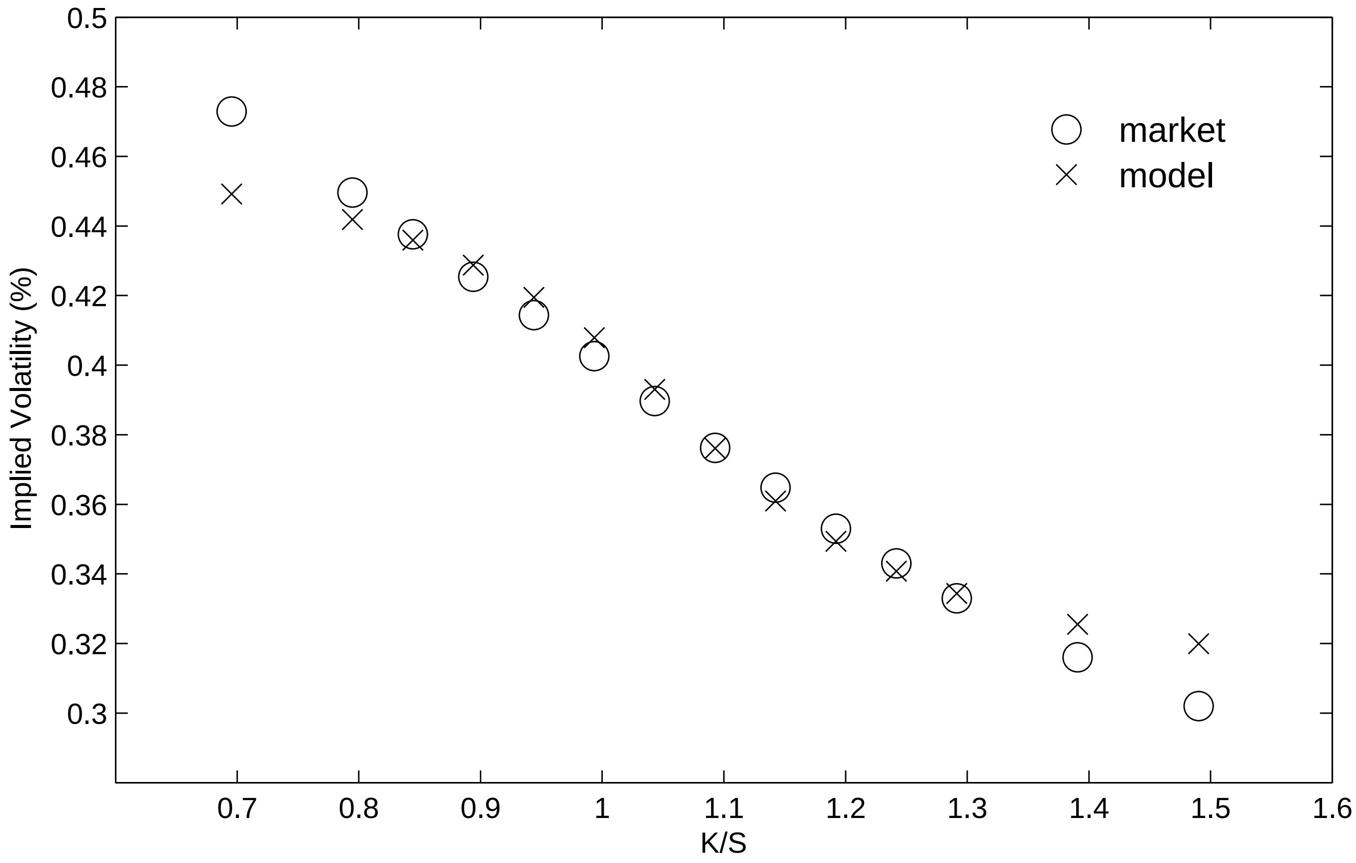
- Need c, β, and δ
- Select params which min SSE between option model prices, market prices

For context, closing level of S&P 500 Index as of May 26, 2009 was 910.33

Estimated parameters: $\hat{c} = 925$, $\hat{\beta} = 1.17$, and $\hat{\delta} = 0.65$.

So market is below threshold

Figure 4: Volatility Skews for Amgen Call Options



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THANK YOU!