Smooth Fit Principle for Impulse Control Problems

Guoliang Wu
UT Austin

Joint work with Mark Davis (Imperial College) and Xin Guo (UC Berkeley)

The Third WCMF, UCSB
November 15, 2009

Outline

Outline

Outline

Impulse Control Problems – Brief Introduction

- No transaction cost / only proportional cost $\rightarrow$ optimal strategy with infinite variation.
- In contrast, assuming fixed cost + proportional cost, strategies with infinitely many transactions within finite time will not be optimal.
- *Fixed Cost* $\rightarrow$ Key characteristics of impulse controls.

Related Work

- Quasi-Variational Inequalities: Caffarelli – Friedman('78, '79), Bensoussan – Lions ('82);
- Cash management: Constantinides – Richard ('78);
- Inventory controls: Harrison – Taylor ('78), Harrison – Sellke – Taylor ('83);
- Portfolio management with transaction cost: Davis – Norman ('90), Korn ('98, '99), Øksendal – Sulem ('02);
- Exchange rates: Jeanblanc-Piqué ('93), Cadenillas – Zapatero ('99).
- Insurance models: Cadenillas et al. ('06);
- Liquidity risk: Ly Vath et al. ('07);
- Irreversible investment: Scheinkman – Zariphopoulou ('01).
- American Options for Jump Diffusions: Bayraktar, Bayraktar – Xing.

Outline

Motivating Example

This example is taken from Constantinides – Richard ('78). Consider the following cash management problem.

- Cash balance on a bank account: $X_t = x + \mu t + \sigma W_t$ (due to a *random* cash demand).
- Holding cost: hX_t if $X_t > 0$. (e.g., opportunity cost).
- Penalty cost: $-pX_t$ if $X_t < 0$.
- The controller decides
  (1) the times $(\tau_1, \tau_2, \dots)$ and (2) the sizes $(\xi_1, \xi_2, \dots)$
  to adjust the cash balance.
- At τ_i , the cash level is adjusted from $X_{\tau_i^-}$ to $X_{\tau_i^-} + \xi_i$, incurring a fixed transaction cost K^+ (or K^-) and a proportional cost $k^+\xi_i$ (or $-k^-\xi_i$).

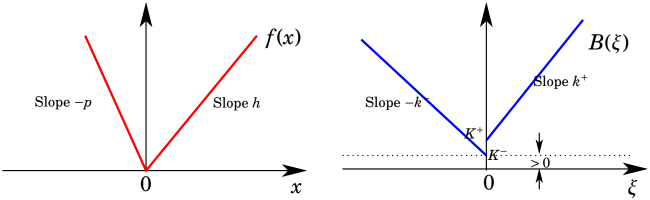
Motivating Example

The goal is to minimize the cost

$$J_x := \mathbb{E}\left(\int_0^\infty e^{-rt} f(X_t)dt + \sum_{i=1}^{\infty} e^{-r\tau_i} B(\xi_i)\right),$$

where

$$f(x) = \begin{cases} hx, & \text{if } x \geq 0 \\ -px, & \text{if } x \leq 0, \end{cases} \qquad B(\xi) = \begin{cases} K^+ + k^+\xi, & \text{if } \xi > 0 \\ K^- - k^-\xi, & \text{if } \xi < 0. \end{cases}$$



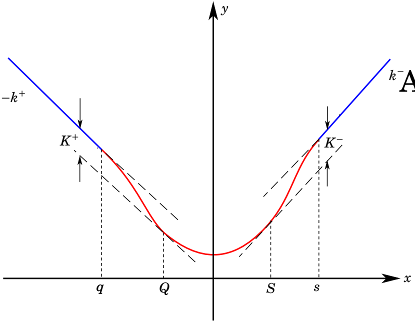
Motivating Example

Questions:

1. Can we find a closed-form solution for the value function
$$u(x) := \inf_{\{\tau_i, \xi_i\}} J_x \quad ?$$
2. Can we find optimal strategies?

Motivating Example – Value Function

Closed-form solution can be obtained by solving the HJB equation and *assuming* the "smooth-fit" principle.



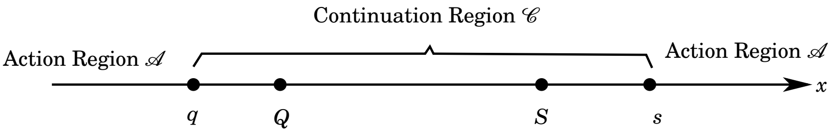
In $\mathscr{C} = (q, s)$,

$$-\frac{\sigma^2}{2}u'' - \mu u' + ru = f(x).$$

And

$$u'(q) = u'(Q) = -k^+$$
$$u'(s) = u'(S) = k^-$$
$$u(q) = u(Q) + K^+ + k^+(Q - q)$$
$$u(s) = u(S) + K^- - k^-(S - s).$$
$$\to q, Q, s, S.$$

Motivating Example – Optimal Strategy



The optimal strategy:

1. If $X_t \in \mathscr{C} = (q, s)$: No action;
2. X_t reaches q (or initial $x \leq q$): raise it to Q immediately;
3. X_t reaches s (or initial $x \geq s$): lower it to S immediately.

It is usually called the "(S, s) policy".

Outline

Why "Smooth Fit Principle"?

1. Getting closed-form solutions relies heavily on the "smooth fit principle".
2. High dimensional problems: "smooth fit" is also important for designing numerical schemes.
3. "Correctness" of the solution is dubious without regularity studies.
4. There are control problems in which "smooth-fit" fail (Guo – Tomecek[3]).

Outline

Outline

The Model

- Probability space: $(\Omega, \mathscr{F}, \mathbb{P})$, with a Brownian motion W.
- Poisson random measure $N(\cdot,\cdot)$ on $\mathbb{R}^+ \times \mathbb{R}^l$.
- W and N are independent.
- Lévy measure $\nu(\cdot) := \mathbb{E}(N(1,\cdot))$.
- Compensated Poisson measure

$$\widetilde{N}(dt,dz) := N(dt,dz) - \nu(dz)dt.$$

- Filtration $\{\mathscr{F}_t\}$ generated by W and N.

The Model

- In the absence of control, the state process $X_t \in \mathbb{R}^n$ follows

$$dX_t = \mu(X_{t^-})dt + \sigma(X_{t^-})dW + \int_{\mathbb{R}^l} j(X_{t^-}, z)\widetilde{N}(dt, dz). \qquad (1)$$

- ***Admissible impulse control*** $V = (\tau_1, \xi_1; \tau_2, \xi_2; \dots)$:
  - $\mathscr{F}_t$-stopping times $\{\tau_i\}_i$ satisfying

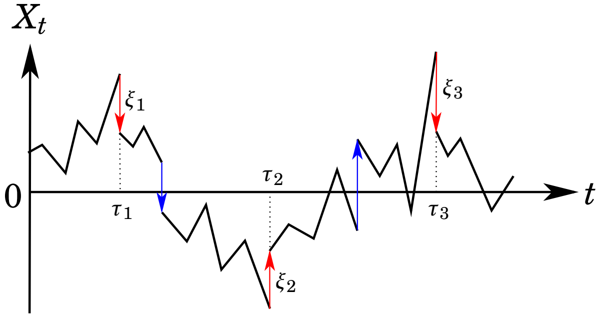
$$\begin{cases} 0 < \tau_1 < \tau_2 < \cdots < \tau_i < \dots, \\ \tau_i \to \infty \text{ as } i \to \infty, \end{cases}$$

  - $\mathscr{F}_{\tau_i}$-measurable r.v.'s $\{\xi_i\}_i$, $\mathbb{R}^n$-valued.

The Model

When $V = (\tau_1, \xi_1; \tau_2, \xi_2; \ldots)$ is adopted, X_t is governed by

$$dX_t = \mu(X_{t^-})dt + \sigma(X_{t^-})dW_t + \int_{\mathbb{R}^l} j(X_{t^-}, z)\widetilde{N}(dt, dz) + \sum_i \delta(t - \tau_i)\xi_i \tag{2}$$



The Model

- Goal: To minimize over all admissible controls

$$J_x[V] := \mathbb{E}_x \left(\int_0^\infty e^{-rt} f(X_t)dt + \sum_{i=1}^{\infty} e^{-r\tau_i} B(\xi_i) \right).$$

- Value function:

$$u(x) := \inf_V J_x[V]. \tag{3}$$

Standing Assumptions

1. $\mu : \mathbb{R}^n \to \mathbb{R}^n, \sigma : \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n, f : \mathbb{R}^n \to \mathbb{R}$ are Lipschitz. $f \geq 0$.
2. The transaction cost $B : \mathbb{R}^n \to \mathbb{R}$ satisfies
   - $B \in C(\mathbb{R}^n \setminus \{0\})$ and $\lim_{|\xi| \to \infty} |B(\xi)| = \infty$.
   - It costs at least $K > 0$ to make a transaction:
     $$\inf_{\xi \in \mathbb{R}^n} B(\xi) =: K > 0.$$
   - It is never optimal to make two transactions simultaneously:
     $$B(\xi_1) + B(\xi_2) \geq B(\xi_1 + \xi_2) + K, \quad \forall \xi_1, \xi_2 \in \mathbb{R}^n.$$
3. The discount factor $r > 0$ sufficiently large.

Our Approach and Tools

- Viscosity solutions
- PDE tools for regularity:
  - Sobolev imbedding:

$$W^{2,p}(\mathcal{O}) \subset C^{1,\alpha}(\mathcal{O}), \alpha = 1 - n/p.$$

  - Elliptic PDE: $-a_{ij}u_{x_i x_j} + b_i u_{x_i} + cu = f$ in $\mathcal{O}$; $u = g$ on $\partial\mathcal{O}$:
    - Schauder's estimates: C^α data $\Rightarrow u \in C^{2,\alpha}$.
    - Calderon-Zygmund estimates: nice coeff. and $f \in L^p \Rightarrow u \in W^{2,p}$.

No Jump Case

To clearly illustrate our methods, let's remove the jump part in the dynamics temporarily.

$$dX_t = \mu(X_{t^-})dt + \sigma(X_{t^-})dW_t + \int_{\mathbb{R}^l} j(X_{t^-}, z)\tilde{N}(dt, dz)$$

$$\downarrow$$

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t$$

Later we will bring this term back.

Hamilton-Jacobi-Bellman Equation

The value function u should "solve" the HJB equation

$$\max\{Lu - f, u - \mathscr{M}u\} = 0 \text{ in } \mathbb{R}^n, \tag{HJB}$$

where

$$Lu(x) = -a_{ij}(x)u_{x_ix_j}(x) - \mu_i(x)u_{x_i}(x) + ru(x), \tag{4}$$

$$\mathscr{M}u(x) = \inf_{\xi \in \mathbb{R}^n}(u(x+\xi) + B(\xi)), \tag{5}$$

and the matrix $A = \left(a_{ij}\right)_{n\times n} = \frac{1}{2}\sigma(x)\sigma(x)^T$.

Heuristic Derivation of HJB Equation

At time $t = 0$, we have two choices:

1. No intervention is optimal: DPP $\Rightarrow Lu = f$. Otherwise $Lu \leq f$.
2. Intervention by an impulse of size ξ^* is optimal:

   $$u(x) = u(x+\xi^*)+B(\xi^*) \rightsquigarrow u(x) = \inf_{\xi\in\mathbb{R}^n} \{u(x+\xi)+B(\xi)\} = \mathscr{M}u(x).$$

   Otherwise $u \leq \mathscr{M}u$.
3. At least one of the equalities holds

   $$\Rightarrow \max\{Lu - f, u - \mathscr{M}u\} = 0.$$

Preliminary Results

1. The value function u is Lipschitz continuous.
2. The operator $\mathscr{M}$ defined by

$$\mathscr{M}u(x) = \inf_{\xi \in \mathbb{R}^n} (u(x+\xi) + B(\xi))$$

   is increasing, concave and preserves Lipschitz continuity and uniform continuity.

Outline

Viscosity Solutions: $F(\nabla^2 u, \nabla u, u, x) = 0$ in $\mathcal{O}$

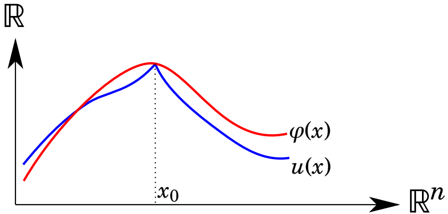
$F(M,p,r,x) \le F(N,p,s,x)$ if $M \ge N$ (matrices ordering) and $r \le s$.

Definition

An upper semi-continuous function u is a viscosity *subsolution* of $F = 0$ in $\mathcal{O}$ provided that for every $\varphi \in C^2(\mathbb{R}^n)$, if $u - \varphi$ has a local maximum at $x_0 \in \mathcal{O}$ and $u(x_0) = \varphi(x_0)$, then

$$F(\nabla^2\varphi(x_0), \nabla\varphi(x_0), \varphi(x_0), x_0) \le 0.$$

Supersolutions are defined similarly.



Viscosity Solutions: $\max\{Lu - f, u - \mathcal{M}u\} = 0$ in $\mathbb{R}^n$

There are (at least) two equivalent definitions:

1. $u \in \mathrm{UC}(\mathbb{R}^n)$ is called a viscosity subsolution of (HJB) provided that for any $\varphi \in C^2(\mathbb{R}^n)$, if $u - \varphi$ has a global maximum at x_0 and $u(x_0) = \varphi(x_0)$, then

$$\max\{L\varphi(x_0) - f(x_0), \varphi(x_0) - \mathcal{M}\varphi(x_0)\} \leq 0.$$

2. $u \in \mathrm{UC}(\mathbb{R}^n)$ is called a viscosity subsolution of (HJB) provided that for any $\varphi \in C^2(\mathbb{R}^n)$, if $u - \varphi$ has a local maximum at x_0 and $u(x_0) = \varphi(x_0)$, then

$$\max\{L\varphi(x_0) - f(x_0), u(x_0) - \mathcal{M}u(x_0)\} \leq 0.$$

The same holds true for supersolutions.

Viscosity Solution Property

The following result is known (Øksendal – Sulem ('04)).

Theorem

The value function u is a viscosity solution of the HJB equation

$$\max\{Lu - f, u - \mathscr{M}u\} = 0 \text{ in } \mathbb{R}^n.$$

Relation with Optimal Stopping Problem

- Optimal stopping problem:

$$v(x) := \inf_{\tau} \mathbb{E}\left(\int_0^{\tau} e^{-rt} f(X_t)dt + e^{-r\tau} g(X_\tau)\right), \tag{6}$$

  subject to $dX_t = \mu(X_t)dt + \sigma(X_t)dW$, $X(0) = x$.
- v is the *unique* uniform continuous viscosity solution of

$$\max\{Lv - f, v - g\} = 0 \text{ in } \mathbb{R}^n.$$

  Uniqueness is established by extending the comparison principle from bounded domains to $\mathbb{R}^n$.

Relation with Optimal Stopping Problem

- Given $w \in \mathrm{UC}(\mathbb{R}^n)$, define (Bensoussan – Lions):

$$\mathcal{T}w(x) := \inf_{\tau} \mathbb{E}\left(\int_0^{\tau} e^{-rt} f(X_t)dt + e^{-r\tau}\mathcal{M}w(X_\tau)\right), \qquad (7)$$

subject to $dX_t = \mu(X_t)dt + \sigma(X_t)dW$, $X(0) = x$.

- $\mathcal{T}w$ is the *unique* viscosity solution of

$$\max\{L(\mathcal{T}w) - f, \mathcal{T}w - \mathcal{M}w\} = 0 \text{ in } \mathbb{R}^n.$$

- If w is a solution of $\max\{Lw - f, w - \mathcal{M}w\} = 0$ in $\mathbb{R}^n$, then

$$\boxed{\mathcal{T}w = w.}$$

Unique Viscosity Solution

Theorem

Assume that there are constants $C, \Lambda > 0$ such that

$$\begin{cases} |\mu(x)| \leq C & \forall x \in \mathbb{R}^n, \\ a_{ij}(x)\eta_i\eta_j \leq \Lambda|\eta|^2 & \forall x, \eta \in \mathbb{R}^n. \end{cases} \tag{8}$$

Then the HJB equation has at most one solution in $\mathrm{BUC}(\mathbb{R}^n)$.

Uniqueness – Sketch Proof

- Suppose $w, v \in \mathrm{BUC}(\mathbb{R}^n)$ are two solutions of (HJB). WLOG, assume $w, v \geq 0$. Then

$$\mathscr{T} w = w, \mathscr{T} v = v.$$

- The operator $\mathscr{T}$ is *increasing* and *concave*.
- The above two properties imply

$$w - v \leq \gamma w \text{ for some } \gamma \geq 0 \Rightarrow w - v \leq \delta \gamma w \text{ for some } \delta < 1.$$

- Starting with $\gamma = 1$, iteration gives $w - v \leq \delta^n \gamma w, \forall n$. Hence $w - v \leq 0$.
- Interchanging w and v, we get $w = v$. $\square$

Outline

Smooth Fit Principle

Theorem (Regularity of Value Function)

Assume that $\sigma \in C^{1,1}$ locally in $\mathbb{R}^n$, and for some $\lambda > 0$,

$$a_{ij}(x)\eta_i\eta_j \geq \lambda|\eta|^2, \quad \forall x, \eta \in \mathbb{R}^n. \text{ (Uniform Ellipticity)}$$

Then for any bounded open set $\mathcal{O} \subset \mathbb{R}^n$ with smooth boundary,

$$u \in W^{2,p}(\mathcal{O}) \quad \forall 1 \leq p < \infty.$$

By **Sobolev imbedding**, $u \in C^1(\mathbb{R}^n)$ and ∇u is in Hölder space C^α for any $\alpha < 1$.
Let us fix an arbitrary bounded open $\mathcal{O}$ with smooth boundary.

Continuation / Action Region / Free Boundary

We define the *continuation region*

$$\mathscr{C} := \{x \in \mathbb{R}^n : u(x) < \mathscr{M}u(x)\}, \tag{9}$$

the *action region*

$$\mathscr{A} := \{x \in \mathbb{R}^n : u(x) = \mathscr{M}u(x)\}, \tag{10}$$

and the *free boundary*

$$\Gamma := \partial\mathscr{A}. \tag{11}$$

Recall that u and $\mathscr{M}u$ are both continuous, so $\mathscr{C}$ is open and $\mathscr{A}$ is closed.

$C^{2,\alpha}$ Regularity in $\mathscr{C}$

Lemma ($C^{2,\alpha}$-Regularity in $\mathscr{C}$)

The value function $u \in C^{2,\alpha}(D)$, *for any* $\alpha \in (0,1)$ *and any compact set* $D \subset \mathscr{C}$, *and it is a classical solution of*

$$Lu(x) - f(x) = 0, \quad x \in \mathscr{C}. \tag{12}$$

This lemma is established using **Schauder's estimates**.

- $\max\{Lu - f, u - \mathscr{M}u\} = 0 \Longrightarrow Lu = f$ in $\mathscr{C} = \{u < \mathscr{M}u\}$.
- $f \in C^{\alpha}(D) \Longrightarrow u \in C^{2,\alpha}(D)$.

Regularity Across Free Boundary – Approach

To obtain regularity of u across the free boundary Γ, we consider again the related optimal stopping problem. In terms of the HJB equations:

$$\text{Impulse:} \quad \max\{Lu - f, u - \mathscr{M}u\} = 0 \text{ in } \mathbb{R}^n. \tag{13}$$

$$\text{Stopping:} \quad \max\{Lv - f, v - g\} = 0 \text{ in } \mathbb{R}^n. \tag{14}$$

What condition should we impose on g to have a “nice” solution v?

Regularity for $\max\{Lv - f, v - g\} = 0$

Lemma

L, f as before. Assume that $g \in C(\mathbb{R}^n)$ and that $\exists \{g^\varepsilon\}_{\varepsilon>0}$ in $C^2(\overline{\mathcal{O}})$ converging uniformly to g in $\overline{\mathcal{O}}$ such that

$$Lg^\varepsilon \geq -M \text{ in } \mathcal{O} \text{ for some } M. \tag{15}$$

If v is a continuous viscosity solution of

$$\max\{Lv - f, v - g\} = 0 \text{ in } \mathbb{R}^n, \tag{16}$$

Then $v \in W^{2,p}(\mathcal{O})$ for any $1 \leq p < \infty$.

Remarks on the Lemma

Suppose $v, g \in C^2(\overline{\mathcal{O}})$ and v solves $\max\{Lv - f, v - g\} = 0$.

- $Lv \le f \le C$ in $\mathcal{O}$.
- If $Lv < f$ at some point $x_0 \in \mathcal{O}$, $v - g$ attains maximum there. By maximum principle, $Lv \ge Lg \ge -M$. Otherwise, $Lv = f \ge -C$.
- We always have $Lv \in L^\infty(\mathcal{O})$.
- By Calderon-Zygmund estimates, $v \in W^{2,p}(\mathcal{O})$.

Observe that in this argument, $Lg \ge -M$ is essential. Unfortunately, we wish to let $g = \mathcal{M}u$ which is not necessary C^2. Hence we approximate g using C^2 functions g^ε with $Lg^\varepsilon \ge -M$.

Proof of Theorem – (1)

To prove the theorem, we apply the above lemma with

$$g = \mathscr{M}u = \inf_{\xi \in \mathbb{R}^n} (u(\cdot + \xi) + B(\xi)) \text{ (Lipschitz)}$$

$$g^\varepsilon = g * \varphi_\varepsilon \in C^\infty,$$

where $\varphi_\varepsilon(x) = \frac{1}{\varepsilon^n}\varphi\left(\frac{|x|}{\varepsilon}\right)$, $\varphi \in C^\infty(\mathbb{R})$ with compact support, $\varphi \geq 0$, and $\int \varphi = 1$.

Then $g^\varepsilon \to g$ uniformly on $\overline{\mathscr{O}}$ and $|\nabla g^\varepsilon| \leq C$. The key is to show that

$$Lg^\varepsilon = -a_{ij} g^\varepsilon_{x_i x_j} - \mu_i g^\varepsilon_{x_i} + r g^\varepsilon \geq -M \text{ in } \mathscr{O},$$

and it suffices to prove

$$a_{ij} g^\varepsilon_{x_i x_j} \leq C \text{ in } \mathscr{O}.$$

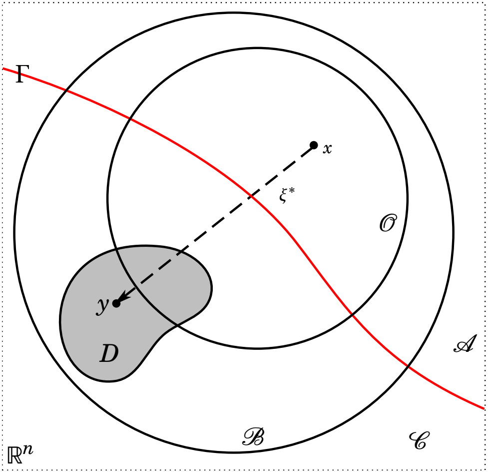
Proof of Theorem – (2)

To estimate $\nabla^2 g^\varepsilon$, consider the second-order difference quotients in the direction $e \in \mathbb{R}^n$ ($|e| = 1$) at $x \in \mathscr{O}$,

$$D^h_{ee} g^\varepsilon(x) := \frac{1}{h^2}\left[g^\varepsilon(x+he) + g^\varepsilon(x-he) - 2g^\varepsilon(x)\right] = \left(D^h_{ee} g\right) * \varphi^\varepsilon.$$

Thus, we seek an upper bound of $D^h_{ee} g(x)$ first.

Proof of Theorem – Figure



$x \in \mathscr{O}$

$y = x + \xi^* \in D$

$u(y) - \mathscr{M}u(y) \leq -K$

$D := \left\{ y \in \mathscr{B} : u(y) \leq \mathscr{M}u(y) - \frac{K}{2} \right\}$

$D^h_{ee} g(x) \leq D^h_{ee} u(y)$

Figure: Proof of Regularity Theorem

Proof of Theorem – (3)

Fix any $x \in \mathcal{O}$ and take a minimizing sequence $\{\xi_k\}$ such that $u(x+\xi_k)+B(\xi_k) \to \mathcal{M}u(x) = g(x)$. Then $\{\xi_k\}$ is bounded. WLOG, $\xi_k \to \xi^*$. Since $B(\xi)+B(\xi') \geq K + B(\xi+\xi')$,

$$\begin{aligned}
\mathcal{M}u(x) &= \inf_{\eta\in\mathbb{R}^n}\{u(x+\xi_k+\eta)+B(\xi_k+\eta)\} \\
&\leq \inf_{\eta\in\mathbb{R}^n}\{u(x+\xi_k+\eta)+B(\eta)\}+B(\xi_k)-K \\
&= \mathcal{M}u(x+\xi_k)+B(\xi_k)-K \\
&= \mathcal{M}u(x+\xi_k)-u(x+\xi_k)+[u(x+\xi_k)+B(\xi_k)]-K.
\end{aligned}$$

Passing to the limit $k\to\infty$, we obtain

$$u(x+\xi^*)-\mathcal{M}u(x+\xi^*) \leq -K.$$

Proof of Theorem – (4)

We can take an open ball $\mathscr{B} \supset \mathscr{O}$ such that

$$x \in \mathscr{O}, u(x+\xi) + B(\xi) \leq \mathscr{M}u(x) + 1 \Longrightarrow x + \xi \in \mathscr{B},$$

since $B(\xi) \to \infty$ as $|\xi| \to \infty$ and $u \geq 0$.
Recall $K = \inf B > 0$. Define

$$D := \left\{ y \in \mathscr{B} : u(y) \leq \mathscr{M}u(y) - \frac{K}{2} \right\}. \tag{17}$$

Then $D \subset \mathscr{C}$ and hence $u \in C^{2,\alpha}(D)$.
Hence $y := x + \xi^*$ is an *interior point* of D.

Proof of Theorem – (5)

Since $\mathscr{M}u(x \pm he) \leq u(x \pm he + \xi_k) + B(\xi_k)$ for all k,

$$
\begin{aligned}
&\mathscr{M}u(x+he) + \mathscr{M}u(x-he) - 2\mathscr{M}u(x) \\
\leq\, & u(x+he+\xi_k) + u(x-he+\xi_k) + 2B(\xi_k) - 2\mathscr{M}u(x) \\
\to\, & u(y+he) + u(y-he) - 2u(y), \quad k \to \infty, \\
\Longrightarrow\, & \boxed{D^h_{ee}g(x) \leq D^h_{ee}u(y)} \leq C_D := \|u\|_{C^2(D)}. \\
\Longrightarrow\, & D^h_{ee}g^{\varepsilon}(x) = \int D^h_{ee}g(x-z)\varphi_{\varepsilon}(z)dz \leq C_D.
\end{aligned}
$$

Sending $h \to 0$,

$$
\begin{aligned}
& e^T\left(\nabla^2 g^{\varepsilon}\right)e \leq C_D \text{ in } \mathscr{O}. \\
\Longrightarrow\, & a_{ij}g^{\varepsilon}_{x_ix_j} = \operatorname{tr}\left[\sigma\sigma^T\left(\nabla^2 g^{\varepsilon}\right)\right] = \operatorname{tr}\left[\sigma^T\left(\nabla^2 g^{\varepsilon}\right)\sigma\right] \leq C \text{ in } \mathscr{O}. \quad \square
\end{aligned}
$$

Outline

Jump Diffusion Model

Let us bring back the jump term. The controlled process obeys

$$dX_t = \mu(X_{t^-})dt + \sigma(X_{t^-})dW_t + \int_{\mathbb{R}^l} j(X_{t^-}, z)\widetilde{N}(dt, dz) + \sum_i \delta(t - \tau_i)\xi_i$$

In addition to the previous conditions, we assume:

1. $|j(x,z) - j(y,z)| \le \rho(z)|x - y|, \forall x, y \in \mathbb{R}^n$ with $\rho(\cdot)$ "nice".
2. $j(x, \cdot) \in L^1(\mathbb{R}^l; \nu), \forall x \in \mathbb{R}^n$.

Remark

The natural condition seems to be $\int (1 \wedge j(x,z)^2)\nu(dz) < \infty$, generalizing the property of ν: $\int (1 \wedge z^2)\nu(dz) < \infty$.
But in the case that $j(x, \cdot)$ is not integrable, the HJB equation is essentially different.

HJB Equation

Theorem

The value function $u(\cdot)$ is a viscosity solution of

$$\max\{\mathscr{L}u - f, u - \mathscr{M}u\} = 0 \; in \; \mathbb{R}^n. \tag{HJB}$$

The only difference is the operator $\mathscr{L}$ which reads

$$\begin{aligned} \mathscr{L}u &= Lu - Iu, \\ Lu(x) &= -a_{ij}(x)u_{x_i x_j}(x) - \left(\mu(x) - \int_{\mathbb{R}^l} j(x,z)\nu(dz)\right) \cdot \nabla u(x) + ru(x), \\ Iu(x) &= \int_{\mathbb{R}^l} [u(x + j(x,z)) - u(x)]\,\nu(dz). \end{aligned}$$

Preliminary Results

We have

1. u and $\mathscr{M}u$ are Lipschitz.
2. $Iu(x) = \int_{\mathbb{R}^l} [u(x+j(x,z)) - u(x)]\,\nu(dz)$ is continuous.

Regularity for Jump Diffusion Model

Theorem

Assume that $\sigma \in C^{1,1}$ locally in $\mathbb{R}^n$ and for some $\lambda > 0$,

$$a_{ij}(x)\eta_i\eta_j \geq \lambda|\eta|^2, \quad \forall x, \eta \in \mathbb{R}^n.$$

Then for any bounded open set $\mathscr{O} \subset \mathbb{R}^n$ and $p < \infty$, we have

$$u \in W^{2,p}(\mathscr{O}).$$

As soon as we have the $C^{2,\alpha}$ regularity in $\mathscr{C}$, the rest of the proof turns out to be the same as the no-jump case.

Key Lemma

Lemma ($C^{2,\alpha}$ Regularity in $\mathscr{C}$)

Assume that $\sigma \in C^1(\mathbb{R}^n)$, then for any compact set $D \subset \mathscr{C}$ and $\alpha \in (0,1)$, we have $u(\cdot) \in C^{2,\alpha}(D)$, and it is a classical solution of

$$\mathscr{L}u - f(x) = 0 \text{ in } \mathscr{C}.$$

- Difference from the no-jump case: The operator $\mathscr{L}$ has an integral term. I.e., in $\mathscr{C}$,

$$\mathscr{L}u = f \Longrightarrow Lu = f + Iu.$$

- Difficulty: Schauder's estimates need $f + Iu \in C^\alpha$. But we don't know Iu is Lipschitz or even Hölder.

Sketch Proof of Key Lemma

The main technique is to “bootstrap”:

1. Iu is continuous, by Calderon-Zygmund estimates

$$Lu = f + Iu \in L^p(D) \Rightarrow u \in W^{2,p}(D).$$

2. By Sobolev imbedding, $u \in W^{2,p}(D) \Rightarrow u \in C^{1,\alpha}(D)$.
3. $u \in C^{1,\alpha}(D)$ implies

$$Iu = \int_{\mathbb{R}^l} [u(\cdot + j(\cdot, z)) - u(\cdot)]\, \nu(dz) \in C^{\alpha}(D).$$

4. Finally, by Schauder estimates,
   $Lu = f + Iu \in C^{\alpha}(D) \Rightarrow u \in C^{2,\alpha}(D)$. □

References

G. M. Constantinides and S. F. Richard, *Existence of optimal simple policies for discounted-cost inventory and cash management in continuous time*, Oper. Res., 26 (1978), pp. 620–636.

M. H. A. Davis, X. Guo, and G. Wu, *Impulse controls of multidimensional jump diffusions*, (preprint).

X. Guo and P. Tomecek, *A class of singular control problems and the smooth fit principle*, SIAM J. Control Optim., 47 (2009), pp. 3076–3099.

X. Guo and G. Wu, *Smooth fit principle for impulse control of multidimensional diffusion processes*, SIAM J. Control Optim., 48 (2009), pp. 594–617.

Thank you!