Smooth Fit Principle for Impulse Control Problems

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Impulse Control Problems – Brief Introduction

- No transaction cost / only proportional cost → optimal strategy with infinite variation.

- In contrast, assuming fixed cost + proportional cost, strategies with infinitely many transactions within finite time will not be optimal.

- *Fixed Cost* → Key characteristics of impulse controls.
Related Work

- Quasi-Variational Inequalities: Caffarelli – Friedman (’78, ’79), Bensoussan – Lions (’82);
- Cash management: Constantinides – Richard (’78);
- Inventory controls: Harrison – Taylor (’78), Harrison – Sellke – Taylor (’83);
- Portfolio management with transaction cost: Davis – Norman (’90), Korn (’98, ’99), Øksendal – Sulem (’02);
- Insurance models: Cadenillas et al. (’06);
- Liquidity risk: Ly Vath et al. (’07);
- Irreversible investment: Scheinkman – Zariphopoulou (’01).
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Motivating Example

This example is taken from Constantinides – Richard (’78). Consider the following cash management problem.

- Cash balance on a bank account: $X_t = x + \mu t + \sigma W_t$ (due to a random cash demand).
- Holding cost: $hX_t$ if $X_t > 0$. (e.g., opportunity cost).
- Penalty cost: $-pX_t$ if $X_t < 0$.
- The controller decides (1) the times ($\tau_1, \tau_2, \ldots$) and (2) the sizes ($\xi_1, \xi_2, \ldots$) to adjust the cash balance.
- At $\tau_i$, the cash level is adjusted from $X_{\tau_i^-}$ to $X_{\tau_i^-} + \xi_i$, incurring a fixed transaction cost $K^+$ (or $K^-$) and a proportional cost $k^+\xi_i$ (or $-k^-\xi_i$).
Motivating Example

The goal is to minimize the cost

$$J_x := \mathbb{E} \left( \int_0^\infty e^{-rt} f(X_t) dt + \sum_{i=1}^\infty e^{-r\tau_i} B(\xi_i) \right),$$

where

$$f(x) = \begin{cases} hx, & \text{if } x \geq 0 \\ -px, & \text{if } x \leq 0, \end{cases} \quad B(\xi) = \begin{cases} K^+ + k^+ \xi, & \text{if } \xi > 0 \\ K^- - k^- \xi, & \text{if } \xi < 0. \end{cases}$$
Motivating Example

Questions:
1. Can we find a closed-form solution for the value function

\[ u(x) := \inf_{\{\tau_i, \xi_i\}} J_x \]

2. Can we find optimal strategies?
Motivating Example – Value Function

Closed-form solution can be obtained by solving the HJB equation and assuming the “smooth-fit” principle.

In $\mathcal{C} = (q,s)$,

$$-\frac{\sigma^2}{2}u'' - \mu u' + ru = f(x).$$

And

$$u'(q) = u'(Q) = -k^+$$
$$u'(s) = u'(S) = k^-$$
$$u(q) = u(Q) + K^+ + k^+(Q - q)$$
$$u(s) = u(S) + K^- - k^-(S - s).$$

$\rightarrow q, Q, s, S$. 
The optimal strategy:

1. If $X_t \in \mathcal{C} = (q,s)$: No action;
2. $X_t$ reaches $q$ (or initial $x \leq q$): raise it to $Q$ immediately;
3. $X_t$ reaches $s$ (or initial $x \geq s$): lower it to $S$ immediately.

It is usually called the "$(S,s)$ policy".
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Why “Smooth Fit Principle”? 

1. Getting closed-form solutions relies heavily on the “smooth fit principle”.
2. High dimensional problems: “smooth fit” is also important for designing numerical schemes.
3. “Correctness” of the solution is dubious without regularity studies.
4. There are control problems in which “smooth-fit” fail (Guo – Tomecek[3]).
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The Model

- Probability space: $(\Omega, \mathcal{F}, \mathbb{P})$, with a Brownian motion $W$.
- Poisson random measure $N(\cdot, \cdot)$ on $\mathbb{R}^+ \times \mathbb{R}^l$.
- $W$ and $N$ are independent.
- Lévy measure $\nu(\cdot) := \mathbb{E}(N(1, \cdot))$.
- Compensated Poisson measure
  \[ \tilde{N}(dt, dz) := N(dt, dz) - \nu(dz)dt. \]
- Filtration $\{\mathcal{F}_t\}$ generated by $W$ and $N$. 
The Model

- In the absence of control, the state process $X_t \in \mathbb{R}^n$ follows

$$dX_t = \mu(X_t^-)dt + \sigma(X_t^-)dW + \int_{\mathbb{R}^l} j(X_t^-, z)\tilde{N}(dt, dz). \quad (1)$$

- **Admissible impulse control** $V = (\tau_1, \xi_1; \tau_2, \xi_2; \ldots)$:
  - $\mathcal{F}_t$-stopping times $\{\tau_i\}_i$ satisfying
    
    $$\begin{cases} 
    0 < \tau_1 < \tau_2 < \cdots < \tau_i < \cdots, \\
    \tau_i \to \infty \text{ as } i \to \infty,
    \end{cases}$$
  
  - $\mathcal{F}_{\tau_i}$-measurable r.v.'s $\{\xi_i\}_i$, $\mathbb{R}^n$-valued.
The Model

When $V = (\tau_1, \xi_1; \tau_2, \xi_2; \ldots)$ is adopted, $X_t$ is governed by

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t + \int_{\mathbb{R}_t} j(X_t, z)\tilde{N}(dt, dz) + \sum_i \delta(t - \tau_i)\xi_i$$

(2)
The Model

- **Goal:** To minimize over all admissible controls

\[
J_x[V] := \mathbb{E}_x \left( \int_0^\infty e^{-rt} f(X_t) dt + \sum_{i=1}^\infty e^{-r\tau_i} B(\xi_i) \right).
\]

- **Value function:**

\[
u(x) := \inf_V J_x[V].
\] (3)
Standing Assumptions

1. \( \mu : \mathbb{R}^n \rightarrow \mathbb{R}^n, \sigma : \mathbb{R}^n \rightarrow \mathbb{R}^n \times \mathbb{R}^n, f : \mathbb{R}^n \rightarrow \mathbb{R} \) are Lipschitz. 
   \( f \geq 0. \)

2. The transaction cost \( B : \mathbb{R}^n \rightarrow \mathbb{R} \) satisfies
   - \( B \in C(\mathbb{R}^n \setminus \{0\}) \) and \( \lim_{|\xi| \to \infty} |B(\xi)| = \infty. \)
   - It costs at least \( K > 0 \) to make a transaction:
     \[
     \inf_{\xi \in \mathbb{R}^n} B(\xi) =: K > 0.
     \]
   - It is never optimal to make two transactions simultaneously:
     \[
     B(\xi_1) + B(\xi_2) \geq B(\xi_1 + \xi_2) + K, \quad \forall \xi_1, \xi_2 \in \mathbb{R}^n.
     \]

3. The discount factor \( r > 0 \) sufficiently large.
Our Approach and Tools

- Viscosity solutions
- PDE tools for regularity:
  - Sobolev imbedding:
    \[ W^{2,p}(\Omega) \subset C^{1,\alpha}(\Omega), \alpha = 1 - n/p. \]
  - Elliptic PDE: \[-a_{ij}u_{x_i x_j} + b_i u_{x_i} + cu = f \text{ in } \Omega; u = g \text{ on } \partial \Omega: \]
    - Schauder’s estimates: \( C^\alpha \) data \( \Rightarrow u \in C^{2,\alpha}. \)
    - Calderon-Zygmund estimates: nice coeff. and \( f \in L^p \Rightarrow u \in W^{2,p}. \)
No Jump Case

To clearly illustrate our methods, let’s remove the jump part in the dynamics temporarily.

\[
dX_t = \mu(X_t^-)dt + \sigma(X_t^-)dW_t + \int_{\mathbb{R}^l} j(X_t^-, z) \tilde{N}(dt, dz)
\]

\[
\downarrow
\]

\[
dX_t = \mu(X_t)dt + \sigma(X_t)dW_t
\]

Later we will bring this term back.
The value function $u$ should “solve” the HJB equation
\[
\max\{Lu - f, u - M u\} = 0 \text{ in } \mathbb{R}^n, \quad \text{(HJB)}
\]
where
\[
Lu(x) = -a_{ij}(x)u_{x_i x_j}(x) - \mu_i(x)u_{x_i}(x) + ru(x), \quad \text{(4)}
\]
\[
M u(x) = \inf_{\xi \in \mathbb{R}^n} (u(x + \xi) + B(\xi)), \quad \text{(5)}
\]
and the matrix $A = (a_{ij})_{n \times n} = \frac{1}{2} \sigma(x)\sigma(x)^T$. 
Heuristic Derivation of HJB Equation

At time $t = 0$, we have two choices:

1. No intervention is optimal: DPP $\Rightarrow Lu = f$. Otherwise $Lu \leq f$.

2. Intervention by an impulse of size $\xi^*$ is optimal:

$$u(x) = u(x + \xi^*) + B(\xi^*) \leadsto u(x) = \inf_{\xi \in \mathbb{R}^n} \{u(x + \xi) + B(\xi)\} = \mathcal{M}u(x).$$

Otherwise $u \leq \mathcal{M}u$.

3. At least one of the equalities holds

$$\Rightarrow \max\{Lu - f, u - \mathcal{M}u\} = 0.$$
Preliminary Results

1. The value function \( u \) is Lipschitz continuous.
2. The operator \( \mathcal{M} \) defined by

\[
\mathcal{M} u(x) = \inf_{\xi \in \mathbb{R}^n} (u(x + \xi) + B(\xi))
\]

is increasing, concave and preserves Lipschitz continuity and uniform continuity.
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Viscosity Solutions: \( F(\nabla^2 u, \nabla u, u, x) = 0 \) in \( \Omega \)

\[ F(M, p, r, x) \leq F(N, p, s, x) \text{ if } M \geq N \text{ (matrices ordering) and } r \leq s. \]

**Definition**

An upper semi-continuous function \( u \) is a viscosity subsolution of \( F = 0 \) in \( \Omega \) provided that for every \( \phi \in C^2(\mathbb{R}^n) \), if \( u - \phi \) has a local maximum at \( x_0 \in \Omega \) and \( u(x_0) = \phi(x_0) \), then

\[ F(\nabla^2 \phi(x_0), \nabla \phi(x_0), \phi(x_0), x_0) \leq 0. \]

Supersolutions are defined similarly.
Viscosity Solutions:
\[ \max\{Lu - f, u - \mathcal{M}u\} = 0 \text{ in } \mathbb{R}^n \]

There are (at least) two equivalent definitions:

1. \( u \in \text{UC}(\mathbb{R}^n) \) is called a viscosity subsolution of (HJB) provided that for any \( \varphi \in C^2(\mathbb{R}^n) \), if \( u - \varphi \) has a global maximum at \( x_0 \) and \( u(x_0) = \varphi(x_0) \), then

\[ \max\{L\varphi(x_0) - f(x_0), \varphi(x_0) - \mathcal{M}\varphi(x_0)\} \leq 0. \]

2. \( u \in \text{UC}(\mathbb{R}^n) \) is called a viscosity subsolution of (HJB) provided that for any \( \varphi \in C^2(\mathbb{R}^n) \), if \( u - \varphi \) has a local maximum at \( x_0 \) and \( u(x_0) = \varphi(x_0) \), then

\[ \max\{L\varphi(x_0) - f(x_0), u(x_0) - \mathcal{M}u(x_0)\} \leq 0. \]

The same holds true for supersolutions.
The following result is known (Øksendal – Sulem ('04)).

**Theorem**

The value function $u$ is a viscosity solution of the HJB equation

$$\max\{Lu - f, u - M u\} = 0 \text{ in } \mathbb{R}^n.$$
Relation with Optimal Stopping Problem

Optimal stopping problem:

\[ v(x) := \inf \mathbb{E} \left( \int_0^\tau e^{-rt} f(X_t) dt + e^{-r\tau} g(X_\tau) \right), \]  

subject to \( dX_t = \mu(X_t) dt + \sigma(X_t) dW \), \( X(0) = x \).

- \( v \) is the unique uniform continuous viscosity solution of

\[ \max\{Lv - f, v - g\} = 0 \text{ in } \mathbb{R}^n. \]

Uniqueness is established by extending the comparison principle from bounded domains to \( \mathbb{R}^n \).
Relation with Optimal Stopping Problem

- Given $w \in \text{UC}(\mathbb{R}^n)$, define (Bensoussan – Lions):

  $$
  \mathcal{T} w(x) := \inf_{\tau} \mathbb{E} \left( \int_0^\tau e^{-rt} f(X_t)dt + e^{-r\tau} M w(X_\tau) \right),
  \tag{7}
  $$

  subject to $dX_t = \mu(X_t)dt + \sigma(X_t)dW$, $X(0) = x$.

- $\mathcal{T} w$ is the unique viscosity solution of

  $$
  \max\{L(\mathcal{T} w) - f, \mathcal{T} w - M w\} = 0 \text{ in } \mathbb{R}^n.
  $$

- If $w$ is a solution of $\max\{L w - f, w - M w\} = 0 \text{ in } \mathbb{R}^n$, then

  $$
  \mathcal{T} w = w.
  $$
Unique Viscosity Solution

Theorem

Assume that there are constants $C, \Lambda > 0$ such that

\[
\begin{align*}
|\mu(x)| & \leq C & \forall x \in \mathbb{R}^n, \\
\alpha_{ij}(x) \eta_i \eta_j & \leq \Lambda |\eta|^2 & \forall x, \eta \in \mathbb{R}^n.
\end{align*}
\]

Then the HJB equation has at most one solution in $\text{BUC}(\mathbb{R}^n)$. 
Suppose \( w, v \in BUC(\mathbb{R}^n) \) are two solutions of (HJB). WLOG, assume \( w, v \geq 0 \). Then

\[ T w = w, \ T v = v. \]

The operator \( T \) is increasing and concave.

The above two properties imply

\[ w - v \leq \gamma w \text{ for some } \gamma \geq 0 \Rightarrow w - v \leq \delta \gamma w \text{ for some } \delta < 1. \]

Starting with \( \gamma = 1 \), iteration gives \( w - v \leq \delta^n \gamma w, \forall n. \) Hence \( w - v \leq 0. \)

Interchanging \( w \) and \( v \), we get \( w = v. \)
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Theorem (Regularity of Value Function)
Assume that $\sigma \in C^{1,1}$ locally in $\mathbb{R}^n$, and for some $\lambda > 0$,

$$a_{ij}(x)\eta_i \eta_j \geq \lambda |\eta|^2, \quad \forall x, \eta \in \mathbb{R}^n. \quad (\text{Uniform Ellipticity})$$

Then for any bounded open set $\mathcal{O} \subset \mathbb{R}^n$ with smooth boundary,

$$u \in W^{2,p}(\mathcal{O}) \quad \forall 1 \leq p < \infty.$$ 

By Sobolev imbedding, $u \in C^1(\mathbb{R}^n)$ and $\nabla u$ is in Hölder space $C^\alpha$ for any $\alpha < 1$.

Let us fix an arbitrary bounded open $\mathcal{O}$ with smooth boundary.
We define the *continuation region*

\[ C := \{ x \in \mathbb{R}^n : u(x) < M u(x) \}, \]  

(9)

the *action region*

\[ A := \{ x \in \mathbb{R}^n : u(x) = M u(x) \}, \]  

(10)

and the *free boundary*

\[ \Gamma := \partial A. \]  

(11)

Recall that \( u \) and \( M u \) are both continuous, so \( C \) is open and \( A \) is closed.
Lemma ($C^{2,\alpha}$-Regularity in $\mathcal{C}$)

The value function $u \in C^{2,\alpha}(D)$, for any $\alpha \in (0,1)$ and any compact set $D \subset \mathcal{C}$, and it is a classical solution of

$$Lu(x) - f(x) = 0, \quad x \in \mathcal{C}.$$  \hspace{1cm} (12)

This lemma is established using Schauder’s estimates.

- $\max\{Lu - f, u - Mu\} = 0 \Rightarrow Lu = f$ in $\mathcal{C} = \{u < Mu\}$.
- $f \in C^\alpha(D) \Rightarrow u \in C^{2,\alpha}(D)$. 
To obtain regularity of $u$ across the free boundary $\Gamma$, we consider again the related optimal stopping problem. In terms of the HJB equations:

\begin{align*}
\text{Impulse:} & \quad \max\{Lu - f, u - Mu\} = 0 \text{ in } \mathbb{R}^n. \\
\text{Stopping:} & \quad \max\{Lv - f, v - g\} = 0 \text{ in } \mathbb{R}^n.
\end{align*} \tag{13, 14}

What condition should we impose on $g$ to have a “nice” solution $v$?
Regularity for \( \max\{Lv - f, v - g\} = 0 \)

**Lemma**

Let \( L, f \) as before. Assume that \( g \in C(\mathbb{R}^n) \) and that \( \exists \{g^\epsilon\}_{\epsilon > 0} \) in \( C^2(\overline{O}) \) converging uniformly to \( g \) in \( \overline{O} \) such that

\[
Lg^\epsilon \geq -M \text{ in } \overline{O} \text{ for some } M. \tag{15}
\]

If \( v \) is a continuous viscosity solution of

\[
\max\{Lv - f, v - g\} = 0 \text{ in } \mathbb{R}^n, \tag{16}
\]

Then \( v \in W^{2,p}(\overline{O}) \) for any \( 1 \leq p < \infty \).
Remarks on the Lemma

Suppose $v, g \in C^2(\bar{\Omega})$ and $v$ solves $\max\{Lv - f, v - g\} = 0$.

- $Lv \leq f \leq C$ in $\Omega$.
- If $Lv < f$ at some point $x_0 \in \Omega$, $v - g$ attains maximum there. By maximum principle, $Lv \geq Lg \geq -M$. Otherwise, $Lv = f \geq -C$.
- We always have $Lv \in L^\infty(\Omega)$.
- By Calderon-Zygmund estimates, $v \in W^{2,p}(\Omega)$.

Observe that in this argument, $Lg \geq -M$ is essential. Unfortunately, we wish to let $g = Mu$ which is not necessary $C^2$. Hence we approximate $g$ using $C^2$ functions $g^\varepsilon$ with $Lg^\varepsilon \geq -M$. 
Proof of Theorem – (1)

To prove the theorem, we apply the above lemma with

\[ g = \mathcal{M}u = \inf_{\xi \in \mathbb{R}^n} (u(\cdot + \xi) + B(\xi)) \text{ (Lipschitz)} \]

\[ g^\varepsilon = g * \varphi_\varepsilon \in C^\infty, \]

where \( \varphi_\varepsilon(x) = \frac{1}{\varepsilon^n} \varphi\left(\frac{|x|}{\varepsilon}\right) \), \( \varphi \in C^\infty(\mathbb{R}) \) with compact support, \( \varphi \geq 0 \), and \( \int \varphi = 1 \).

Then \( g^\varepsilon \to g \) uniformly on \( \overline{\mathcal{O}} \) and \( |\nabla g^\varepsilon| \leq C \). The key is to show that

\[ Lg^\varepsilon = -a_{ij}g^\varepsilon_{x_i x_j} - \mu_i g^\varepsilon_{x_i} + rg^\varepsilon \geq -M \text{ in } \mathcal{O}, \]

and it suffices to prove

\[ a_{ij}g^\varepsilon_{x_i x_j} \leq C \text{ in } \mathcal{O}. \]
Proof of Theorem – (2)

To estimate $\nabla^2 g^\varepsilon$, consider the second-order difference quotients in the direction $e \in \mathbb{R}^n \ (|e| = 1)$ at $x \in \mathcal{O}$,

$$D^h_{ee}g^\varepsilon(x) := \frac{1}{h^2} \left[ g^\varepsilon(x + he) + g^\varepsilon(x - he) - 2g^\varepsilon(x) \right] = \left( D^h_{ee}g \right) \ast \varphi^\varepsilon.$$ 

Thus, we seek an upper bound of $D^h_{ee}g(x)$ first.
Proof of Theorem – Figure

Figure: Proof of Regularity Theorem

\[ x \in \mathcal{O} \]
\[ y = x + \xi^* \in D \]
\[ u(y) - M u(y) \leq -K \]
\[ D := \left\{ y \in \mathcal{B} : u(y) \leq M u(y) - \frac{K}{2} \right\} \]
\[ D_{ee}^h g(x) \leq D_{ee}^h u(y) \]
Proof of Theorem – (3)

Fix any $x \in \mathcal{O}$ and take a minimizing sequence $\{\xi_k\}$ such that $u(x + \xi_k) + B(\xi_k) \rightarrow \mathcal{M} u(x) = g(x)$. Then $\{\xi_k\}$ is bounded. WLOG, $\xi_k \rightarrow \xi^*$. Since $B(\xi) + B(\xi') \geq K + B(\xi + \xi')$,

$$
\mathcal{M} u(x) = \inf_{\eta \in \mathbb{R}^n} \{u(x + \xi_k + \eta) + B(\xi_k + \eta)\} \\
\leq \inf_{\eta \in \mathbb{R}^n} \{u(x + \xi_k + \eta) + B(\eta)\} + B(\xi_k) - K \\
= \mathcal{M} u(x + \xi_k) + B(\xi_k) - K \\
= \mathcal{M} u(x + \xi_k) - u(x + \xi_k) + [u(x + \xi_k) + B(\xi_k)] - K.
$$

Passing to the limit $k \rightarrow \infty$, we obtain

$$
u(x + \xi^*) - \mathcal{M} u(x + \xi^*) \leq -K.$$

Proof of Theorem – (4)

We can take an open ball \( \mathcal{B} \supset \mathcal{O} \) such that

\[
x \in \mathcal{O}, u(x + \xi) + B(\xi) \leq M u(x) + 1 \implies x + \xi \in \mathcal{B},
\]

since \( B(\xi) \to \infty \) as \( |\xi| \to \infty \) and \( u \geq 0 \).

Recall \( K = \inf B > 0 \). Define

\[
D := \left\{ y \in \mathcal{B} : u(y) \leq M u(y) - \frac{K}{2} \right\}.
\]  (17)

Then \( D \subset \mathcal{C} \) and hence \( u \in C^{2,\alpha}(D) \).

Hence \( y := x + \xi^* \) is an interior point of \( D \).
Proof of Theorem – (5)

Since $\mathcal{M}u(x \pm he) \leq u(x \pm he + \xi_k) + B(\xi_k)$ for all $k$,

$$
\mathcal{M}u(x + he) + \mathcal{M}u(x - he) - 2\mathcal{M}u(x) \\
\leq u(x + he + \xi_k) + u(x - he + \xi_k) + 2B(\xi_k) - 2\mathcal{M}u(x) \\
\rightarrow u(y + he) + u(y - he) - 2u(y), \quad k \rightarrow \infty,
$$

$$
\implies D_{ee}^h g(x) \leq D_{ee}^h u(y) \leq C_D := \|u\|_{C^2(D)}.
$$

$$
\implies D_{ee}^h g^\varepsilon(x) = \int D_{ee}^h g(x - z) \varphi_\varepsilon(z)dz \leq C_D.
$$

Sending $h \rightarrow 0$,

$$
e^T (\nabla^2 g^\varepsilon) e \leq C_D \text{ in } \mathcal{O}.
$$

$$
\implies a_{ij}g_{x_i x_j}^\varepsilon = \text{tr} \left[ \sigma \sigma^T (\nabla^2 g^\varepsilon) \right] = \text{tr} \left[ \sigma^T (\nabla^2 g^\varepsilon) \sigma \right] \leq C \text{ in } \mathcal{O}. \quad \square
$$
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Let us bring back the jump term. The controlled process obeys

\[
dX_t = \mu(X_t^-)dt + \sigma(X_t^-)dW_t + \int_{\mathbb{R}^l} j(X_t^-, z)\tilde{N}(dt, dz) + \sum_i \delta(t - \tau_i)\xi_i
\]

In addition to the previous conditions, we assume:

1. \(|j(x, z) - j(y, z)| \leq \rho(z)|x - y|, \forall x, y \in \mathbb{R}^n\) with \(\rho(\cdot)\) “nice”.
2. \(j(x, \cdot) \in L^1(\mathbb{R}^l; \nu), \forall x \in \mathbb{R}^n\).

Remark
The natural condition seems to be \(\int (1 \wedge j(x, z)^2)\nu(dz) < \infty\), generalizing the property of \(\nu\): \(\int (1 \wedge z^2)\nu(dz) < \infty\).
But in the case that \(j(x, \cdot)\) is not integrable, the HJB equation is essentially different.
The value function $u(\cdot)$ is a viscosity solution of

$$\max\{Lu - f, u - Mu\} = 0 \text{ in } \mathbb{R}^n.$$  \hspace{1cm} (HJB)

The only difference is the operator $\mathcal{L}$ which reads

$$\mathcal{L}u = Lu - Iu,$$

$$Lu(x) = -a_{ij}(x)u_{x_i x_j}(x) - \left(\mu(x) - \int_{\mathbb{R}^l} j(x, z)\nu(dz)\right) \cdot \nabla u(x) + ru(x),$$

$$Iu(x) = \int_{\mathbb{R}^l} [u(x + j(x, z)) - u(x)]\nu(dz).$$
Preliminary Results

We have

1. $u$ and $Mu$ are Lipschitz.
2. $Iu(x) = \int_{\mathbb{R}^l} [u(x + j(x,z)) - u(x)]\nu(dz)$ is continuous.
Regularity for Jump Diffusion Model

Theorem

Assume that $\sigma \in C^{1,1}$ locally in $\mathbb{R}^n$ and for some $\lambda > 0$,

$$a_{ij}(x)\eta_i \eta_j \geq \lambda |\eta|^2, \quad \forall x, \eta \in \mathbb{R}^n.$$ 

Then for any bounded open set $\Theta \subset \mathbb{R}^n$ and $p < \infty$, we have

$$u \in W^{2,p}(\Theta).$$

As soon as we have the $C^{2,\alpha}$ regularity in $\mathcal{C}$, the rest of the proof turns out to be the same as the no-jump case.
Key Lemma

Lemma ($C^{2,\alpha}$ Regularity in $C$)
Assume that $\sigma \in C^1(\mathbb{R}^n)$, then for any compact set $D \subset C$ and $\alpha \in (0, 1)$, we have $u(\cdot) \in C^{2,\alpha}(D)$, and it is a classical solution of

$$Lu - f(x) = 0 \text{ in } C.$$ 

- Difference from the no-jump case: The operator $L$ has an integral term. I.e., in $C$,

$$Lu = f \implies Lu = f + Iu.$$ 

- Difficulty: Schauder’s estimates need $f + Iu \in C^\alpha$. But we don’t know $Iu$ is Lipschitz or even Hölder.
Sketch Proof of Key Lemma

The main technique is to “bootstrap”:

1. $Iu$ is continuous, by Calderon-Zygmund estimates

$$Lu = f + Iu \in L^p(D) \Rightarrow u \in W^{2,p}(D).$$

2. By Sobolev imbedding, $u \in W^{2,p}(D) \Rightarrow u \in C^{1,\alpha}(D)$.

3. $u \in C^{1,\alpha}(D)$ implies

$$Iu = \int_{\mathbb{R}^l} [u(\cdot + j(\cdot, z)) - u(\cdot)] \nu(dz) \in C^{\alpha}(D).$$

4. Finally, by Schauder estimates,

$$Lu = f + Iu \in C^{\alpha}(D) \Rightarrow u \in C^{2,\alpha}(D).$$
References


Thank you!