On $\lambda$-Quantile Dependent Convex Risk Measures

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Summary of this talk:

- Define the class of $\lambda$-quantile dependent convex measures of risk.
- Define the $\lambda$-quantile dependent Fatou property.
- Give the robust representation of this class of risk measures.
- Example: the Weighted VaR.
• Measures of risk: capital requirements can be added to a financial position to make it acceptable.

• Axioms of convex measures of risk: for $X, Y \in \mathcal{X}$,

  - Monotonicity: If $X \leq Y$, then $\rho(X) \geq \rho(Y)$.
  
  - Cash invariance: If $m \in \mathbb{R}$, then $\rho(X + m) = \rho(X) - m$.
  
  - Convexity: $\rho(\alpha X + (1 - \alpha)Y) \leq \alpha \rho(X) + (1 - \alpha)\rho(Y)$, $\alpha \in [0, 1]$.
  
  $\rho$ is coherent if in addition:
  
  - Positive homogeneity: $\rho(\lambda X) = \lambda \rho(X)$, for all $\lambda \geq 0$ and $X \in \mathcal{X}$. 
Robust Representation of Convex Measures of Risk:

Let \((\Omega, \mathcal{F}, P)\) be a given probability space. Consider a proper convex measure of risk \(\rho : L^p \to \mathbb{R} \cup \infty, 1 \leq p \leq \infty\).

If \(\rho\) is lower semicontinuous, it is well-known that \(\rho\) has following robust representation:

\[
\rho(X) = \sup_{Q \in \mathcal{Q}} (\mathbb{E}_Q[-X] - \rho^*(Q)),
\]

with \(\mathcal{Q} = \{Q\text{ probability measures : } Q \ll P\}\) and \(\rho^*(Q)\) the Fenchel-Legendre transformation of \(\rho\).

If \(\rho\) is coherent, then \(\rho(X) = \sup_{Q \in \check{\mathcal{Q}}} \mathbb{E}_Q[-X]\) for some set \(\check{\mathcal{Q}} \subset \mathcal{Q}\).

Key point to the representation: the lower semicontinuity \(\Leftrightarrow\) Fatou property.
Literatures on Fatou property and robust representation of $\rho$:

• Delbaen (2000): Coherent real-valued $\rho$ on $L^\infty$.

The Fatou property: $(X_n) \subset L^\infty$, $|X_n| \leq C$, then

$X_n \xrightarrow{P} X \in L^\infty$ P-a.s., implies $\rho(X) \leq \lim \inf_{n \to \infty} \rho(X_n)$.

• Föllmer and Schied (2004): Convex real-valued $\rho$ on $L^\infty$.

• Biagini and Frettelli (2009): Convex proper $\rho$ on $L^p$, $1 \leq p \leq \infty$.

Fatou property: $(X_n) \subset L^p$, $1 \leq p \leq \infty$, $|X_n| \leq Y$ P-a.s., $Y \in L^p$, then

$X_n \xrightarrow{P} X$ P-a.s. for some $X \in L^p$ implies $\rho(X) \leq \lim \inf_{n \to \infty} \rho(X_n)$.

• Kaina and Rüschendorf (2009): robust representation of a convex proper $\rho$ on $L^p$, $1 \leq p \leq \infty$. 
Some concerns:

- The Fatou property is the key point $\rho$ to be representable. However, the boundedness of the sequence $X_n$ is in practice not easy to check.

- In practice, often only the loss of a financial position up to some fixed level is concerned.

- If a convex measure of risk depends only on the left tail of the random variables, would the Fatou property be weakened and easier to check?

Answer: Yes!
Examples:

- Value-at-Risk: $VaR_\lambda(X) := -q^+_X(\lambda) = q^-_X(1 - \lambda)$

  Not a convex measure of risk.

- Conditional VaR: $CVaR_\lambda(X) := \frac{1}{\lambda} \int_0^\lambda VaR_\gamma(X) d\gamma = -\frac{1}{\lambda} \int_0^\lambda q^+_X(t) dt$

  A coherent measure of risk.

- Weighted VaR: $\rho_\mu(X) := \int_{[0,1]} CVaR_\gamma \mu(d\gamma) = - \int_0^1 q_X(t) \phi(t) dt$

  with $\mu$ a probability measure on $[0, 1]$ and $\phi(t) := \int_{(t, 1]} \frac{1}{s} \mu(ds)$.

  A coherent measure of risk.
Definition 0.1 A convex measure of risk $\rho : L^p \to \mathbb{R} \cup \infty$ is $\lambda$-quantile dependent, if

$$X \mathbb{1}_{\{X \leq q_X^+(\lambda)\}} = Y \mathbb{1}_{\{Y \leq q_Y^+(\lambda)\}} \quad \mathbb{P} - \text{a.s. implies } \rho(X) = \rho(Y),$$

i.e., $\rho$ depends on the the random variables only up to their $\lambda$-quantiles.

Reward: weaker Fatou property required for the representation.
Fatou property of a $\lambda$-quantile dependent risk measure:

**Definition 0.2 ($\lambda$-quantile Fatou property)**

For any sequence $(X_n) \subset L^p$ such that $q_{X_n}^+(\lambda) \leq c_\lambda$, for some $c_\lambda \in \mathbb{R}$ and all $n \in \mathbb{N}$,

$$X_n \to X \text{ P-a.s. for some } X \in L^p \text{ implies } \rho(X) \leq \liminf_{n \to \infty} \rho(X_n).$$

Comparison:

Delbaen (2000): $|X_n| \leq C$,

Biagini and Frettelli (2009): $|X_n| \leq Y$.

So: $|X_n| \leq C \Rightarrow |X_n| \leq Y \Rightarrow q_{X_n}^+(\lambda) \leq c_\lambda$.

If only the losses of the financial positions are considered, then 0 will be a natural upper bound of the quantiles.
Robust representation of $\lambda$-quantile dependent convex measures of risk:

**Theorem 0.3** Let $\rho : L^p \to \mathbb{R} \cup \infty$, $1 \leq p \leq \infty$, be a proper $\lambda$-quantile dependent convex measure of risk, then the following are equivalent:

1. $\rho$ is $\sigma(L^p,(L^p)'$)-lower semicontinuous.

2. $\rho(X) = \sup_{Q \in \mathcal{Q}_p} (\mathbb{E}_Q[-X] - \rho^*(Q))$,
   with $\rho^*$ the Fenchel-Legendre transformation of $\rho$ and
   \[ \mathcal{Q}_p = \{Q \text{ probability measures} : Q \ll P, \frac{dQ}{dP} \in (L^p)\}' \].

3. $\rho$ is continuous from above.

4. $\rho$ has the $\lambda$-quantile Fatou property.
Sketch of Proof:

“1⇒2⇒3”: see Theorem 4.31 of Föllmer and Schied (2004) for $L^\infty$ case or Theorem 3.3 of Kaina and Rüschendorf (2009) for $L^p$ case.

“3⇒ 4”: Continuous from above⇒ $\rho$ has Fatou property (BF(2009))

⇒ $\rho$ has $\lambda$-quantile dependent Fatou property.

“4⇒1”: Show that $C := \{\rho < c\}$ is weakly closed. Equivalently, show $C_r := C \cap \{X \in L^p : \|X\|_p \leq r\}$ is weakly closed for all $r > 0$.

For $(X_n) \subset C_r$ s.t. $X_n \to X$ in $L^p$-norm, ∃ subsequence $X_{n_k}$ s.t. $X_{n_k} \to X$ $P$-a.s.

⇒ $q^+_{X_{n_k}} (\lambda)$ is uniformly bounded⇒ $\rho(X) \leq \lim \inf_{n \to \infty} \rho(X_n) \leq c$,

⇒ $X \in C_r$ and $C_r$ is strongly closed ⇒ $C_r$ is weakly closed.
An Example: The $\lambda$-quantile dependent Weighted $VaR$.

**Definition 0.4** $\rho_{\mu, \lambda} : L^p \to \mathbb{R} \cup \infty$, $1 \leq p \leq \infty$, is defined as

$$\rho_{\mu, \lambda}(X) = \int_{[0, \lambda]} CVaR_{\gamma}(X) \mu(d\gamma) = -\int_0^\lambda q_X(t)\phi(t)dt = \int_0^\lambda q_X(t)q_{\nu_\phi}(t)dt.$$  

where $\mu$ is a probability measure on $[0, 1]$ s.t. $\mu((\lambda, 1]) = 0$, and assume $\mu(\{0\}) = 0$.

And $-\phi(t) = \int_{(t, \lambda]} \frac{1}{s} \mu(s)$ is monotone increasing on $[0, \lambda]$, it can be viewed as a quantile function of a probability distribution measure $\nu_\phi$ defined as $\nu_\phi([-\phi(0), -\phi(t)]) := t$, and $\nu_\phi(0) = 1 - \lambda$.

Then $q_{\nu_\phi}(t) := -\phi(t)$, $\forall t \in [0, \lambda]$.

Notice that $\int_0^\lambda q_{\nu_\phi}(t)dt = -1$. 
\( \rho_{\mu,\lambda} \) is \( \lambda \)-quantile law invariant:

**Definition 0.5** A convex risk measure \( \rho : L^p \to \mathbb{R} \cup \infty \) is \( \lambda \)-quantile law invariant, if for any \( X, Y \in L^p \),

\[
X 1_{\{X \leq q_X^+(\lambda)\}} \quad \text{and} \quad Y 1_{\{Y \leq q_Y^+(\lambda)\}} \quad \text{have same distribution implies} \quad \rho(X) = \rho(Y).
\]

Recall the definition of \( \lambda \)-quantile dependent convex measure of risk:

\( \rho \) is \( \lambda \)-quantile dependent, if for any \( X, Y \in L^p \),

\[
X 1_{\{X \leq q_X^+(\lambda)\}} = Y 1_{\{Y \leq q_Y^+(\lambda)\}} \quad \mathbb{P} - \text{a.s. implies} \quad \rho(X) = \rho(Y),
\]
The document discusses the \( \lambda \)-quantile uniform preference (second order stochastic dominance) of two probability distribution measures \( \mu \) and \( \nu \):

**Definition 0.6** Let \( \mu, \nu \) be two probability distribution measures. \( \mu \) is \( \lambda \)-quantile uniformly preferred over \( \nu \) if for any "\( \lambda \)-quantile utility function" \( u \) defined as \( u(x) = u_0(x)I\{x \leq q_\nu(\lambda)\} + u_0(q_\nu(\lambda))I\{x > q_\nu(\lambda)\} \) with \( u_0 \) a utility function, the following is true:

\[
\int_0^\lambda u \, d\mu \geq \int_0^\lambda u \, d\nu.
\]

**Proposition 0.7**

\( \mu \trianglerighteq_{uni(\lambda)} \nu \iff \int_0^t q_\mu(s) \, ds \geq \int_0^t q_\nu(s) \, ds, \forall t \in [0, \lambda] \)

\[
\iff \int_0^\lambda h(t) q_\mu(t) \, dt \geq \int_0^\lambda h(t) q_\nu(t) \, dt, \forall \text{ decreasing } h : [0, \lambda] \rightarrow \mathbb{R}^+.
\]
Robust representation of $\rho_{\mu,\lambda}$:

Recall

$$\rho_{\mu,\lambda}(X) = \int_{[0,\lambda]} CVaR_\gamma(X) \mu(d\gamma) = -\int_0^\lambda q_X(t)\phi(t)dt = \int_0^\lambda q_X(t)q_{\nu_\phi}(t)dt.$$ 

**Lemma 0.8** Define

$$\Phi := \left\{ \nu : \nu \text{ distribution measure such that } \nu \succneq_\text{uni}(\lambda) \nu_\phi \text{ and } \int_0^\lambda q_\nu(t)dt = -1 \right\}.$$ 

Then for $X \in L^p$,

$$\rho_{\mu,\lambda}(X) = \max_{\nu \in \Phi} \int_0^\lambda q_X(t)q_\nu(t)dt.$$ 

The maximum is obtained by taking $\tilde{\nu} \in \Phi$ s.t. $q_{\tilde{\nu}} = q_{\nu_\phi}$. 
Theorem 0.9 (Robust representation of $\rho_{\mu,\lambda}$)

For $X \in L^p$, $1 \leq p \leq \infty$,

$$\rho_{\mu,\lambda}(X) = \max_{Q \in \mathcal{Q}} \mathbb{E}_Q[-X],$$

with $\mathcal{Q} :=$

$$\left\{ Q \text{ probability measure } : Q \ll P, \nu \frac{dQ}{dP} \succeq \nu_\phi \text{ and } \int_0^\lambda q \nu \frac{dQ}{dP}(t) dt = -1 \right\}.$$

The maximum is obtained by choosing $Q_X \in \mathcal{Q}$ such that $\frac{dQ_X}{dP} = f(X)$, where the decreasing function $f$ is given by:

$$f(x) = \begin{cases} \phi(F_X(x)) & \text{ if } x \text{ is a continuity point of } F_X, \\ \frac{1}{F_X(x) - F_X(x^-)} \int_{F_X(x^-)}^{F_X(x)} \phi(t) dt & \text{ if } x \text{ is a discrete point of } F_X, \end{cases}$$

for $F_X(x) \leq \lambda$ and $f(x) = 0$, otherwise.
Example: $CVaR_\lambda$ for $\lambda \in (0, 1)$.

Take $\mu(ds) = \mathbb{I}_{\{\lambda\}}(ds)$, $CVaR_\lambda(X)$ is a special case of $\rho_{\mu,\lambda}(X)$.

It is well known that $CVaR_\lambda(X) = -\frac{1}{\lambda} \int_0^\lambda q_X(t)dt = \sup_{Q \in Q_\lambda} \mathbb{E}_Q[-X]$, where

$$Q_\lambda = \left\{ Q \text{ probability measure : } Q \ll P, \quad \frac{dQ}{dP} \leq \frac{1}{\lambda} \quad P - a.s. \right\}.$$ 

The set $Q_\lambda$ coincides with the set $Q$ defined in the Theorem:

$$Q = \left\{ Q \text{ probability measure : } Q \ll P, \nu_{\frac{dQ}{dP}} \geq \nu_\phi, \quad \int_0^\lambda q_\nu \frac{dQ}{dP}(t)dt = -1 \right\}.$$
To verify, consider a $Q \in \mathcal{Q}$. Recall that

$$
\rho_{\mu,\lambda}(X) = \int_{[0,\lambda]} \text{CVaR}_\gamma(X) \mu(\mathrm{d}\gamma) = -\int_0^\lambda q_X(t)\phi(t)\mathrm{d}t = \int_0^\lambda q_X(t)q_{\nu_{\phi}}(t)\mathrm{d}t,
$$

where $\nu_{\phi}([-\phi(0), -\phi(t)]) := t$, $q_{\nu_{\phi}}(t) = -\phi(t)$, $\forall t \in [0, \lambda]$.

In this case, $-\phi(t) = -\int_{(t,1]} \frac{1}{s} \mu(\mathrm{d}s) = -\frac{1}{\lambda} \mathbb{I}_{[0,\lambda]}(t)$ and $q_{\nu_{\phi}}(t) = -\phi(t)$.

$$
\nu_{-\frac{\mathrm{d}Q}{\mathrm{d}P}} \succcurlyeq_{\text{uni}(\lambda)} \nu_{\phi} \iff \int_0^t q_{-\nu_{\frac{\mathrm{d}Q}{\mathrm{d}P}}}(s)\mathrm{d}s \geq \int_0^\lambda q_{\nu_{\phi}}(t)\mathrm{d}t = \int_0^t -\frac{1}{\lambda} \mathbb{I}_{[0,\lambda]}(s)\mathrm{d}s.
$$

If there is a $s \in [0, \lambda]$ s.t. $\frac{\mathrm{d}Q}{\mathrm{d}P}(s) > \frac{1}{\lambda}$, then $q_{-\nu_{\frac{\mathrm{d}Q}{\mathrm{d}P}}}(s) < -\frac{1}{\lambda}$, and since $q_{-\nu_{\frac{\mathrm{d}Q}{\mathrm{d}P}}}(t)$ is monotone increasing,

$$
-1 = \int_0^\lambda q_{-\nu_{\frac{\mathrm{d}Q}{\mathrm{d}P}}}(t)\mathrm{d}t = \int_0^s q_{-\nu_{\frac{\mathrm{d}Q}{\mathrm{d}P}}}(t)\mathrm{d}t + \int_s^\lambda q_{-\nu_{\frac{\mathrm{d}Q}{\mathrm{d}P}}}(t)\mathrm{d}t < -\frac{s}{\lambda} - \frac{\lambda-s}{\lambda} = -1.
$$
If $Q \in Q_\lambda$. Then $\frac{dQ}{dP} \leq \frac{1}{\lambda} P$-a.s. implies

$$\int_0^t q_\nu \frac{dQ}{dP}(s)ds \geq -\frac{t}{\lambda} = \int_0^t -\frac{1}{\lambda} \mathbb{I}_{[0,\lambda]}(s)ds = \int_0^t q_{\nu\phi}(s)ds,$$

for all $t \in [0, \lambda]$. So the distribution measure of $-\frac{dQ}{dP}$ is $\lambda$-quantile uniformly preferred over $\nu_{\phi}$, and it is also true that

$$\int_0^\lambda q_\nu \frac{dQ}{dP}(t)dt = \mathbb{E}[-\frac{dQ}{dP}] = -1.$$
THANK YOU!