On λ-Quantile Dependent Convex Risk Measures

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Summary of this talk:

- Define the class of λ-quantile dependent convex measures of risk.
- Define the λ-quantile dependent Fatou property.
- Give the robust representation of this class of risk measures.
- Example: the Weighted VaR.

- Measures of risk : capital requirements can be added to a financial position to make it acceptable.

- Axioms of convex measures of risk: for $X, Y \in \mathcal{X}$,

  – Monotonicity: If $X \leq Y$, then $\rho(X) \geq \rho(Y)$.

  – Cash invariance: If $m \in \mathbb{R}$, then $\rho(X + m) = \rho(X) - m$.

  – Convexity: $\rho(\alpha X + (1-\alpha)Y) \leq \alpha\rho(X) + (1-\alpha)\rho(Y)$, $\alpha \in [0, 1]$.

  ρ is **coherent** if in addition:

  – Positive homogeneity: $\rho(\lambda X) = \lambda\rho(X), \quad \text{for all } \lambda \geq 0 \text{ and } X \in \mathcal{X}$.

Robust Representation of Convex Measures of Risk:

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a given probability space. Consider a proper convex measure of risk $\rho : L^p \longrightarrow \mathbb{R} \cup \infty$, $1 \leq p \leq \infty$.

If ρ is lower semicontinuous, it is well-known that ρ has following robust representation:

$$\rho(X) = \sup_{\mathbf{Q} \in \mathcal{Q}} (\mathbb{E}_{\mathbf{Q}}[-X] - \rho^*(Q)),$$

with $\mathcal{Q} = \{\mathbf{Q}$ probability measures : $\mathbf{Q} \ll \mathbf{P}\}$ and $\rho^*(\mathbf{Q})$ the Fenchel-Legendre transformation of ρ.

If ρ is coherent, then $\rho(X) = \sup_{\mathbf{Q} \in \tilde{\mathcal{Q}}} \mathbb{E}_{\mathbf{Q}}[-X]$ for some set $\tilde{\mathcal{Q}} \subset \mathcal{Q}$.

Key point to the representation: the lower semicontinuity $\Leftrightarrow$ Fatou property.

Literatures on Fatou property and robust representation of ρ:

- Delbaen (2000): Coherent real-valued ρ on L^∞.

  The Fatou property: $(X_n) \subset L^\infty$, $|X_n| \leq C$, then

  $X_n \to X \in L^\infty$ $\mathbf{P}$-a.s., implies $\rho(X) \leq \liminf_{n\to\infty} \rho(X_n)$.

- Föllmer and Schied (2004): Convex real-valued ρ on L^∞.

- Biagini and Frettelli (2009): Convex proper ρ on L^p, $1 \leq p \leq \infty$.

  Fatou property: $(X_n) \subset L^p$, $1 \leq p \leq \infty$, $|X_n| \leq Y$ $\mathbf{P}$-a.s., $Y \in L^p$, then

  $X_n \to X$ $\mathbf{P}$-a.s. for some $X \in L^p$ implies $\rho(X) \leq \liminf_{n\to\infty} \rho(X_n)$.

- Kaina and Rüschendorf (2009): robust representation of a convex proper

  ρ on L^p, $1 \leq p \leq \infty$.

Some concerns:

- The Fatou property is the key point ρ to be representable. However, the boundedness of the sequence X_n is in practice not easy to check.
- In practice, often only the loss of a financial position up to some fixed level is concerned.
- If a convex measure of risk depends only on the left tail of the random variables, would the Fatou property be weakened and easier to check?

Answer: Yes!

Examples:

- Value-at-Risk: $VaR_\lambda(X) := -q_X^+(\lambda) = q_{-X}^-(1-\lambda)$

  Not a convex measure of risk.

- Conditional VaR: $CVaR_\lambda(X) := \frac{1}{\lambda}\int_0^\lambda VaR_\gamma(X)d\gamma = -\frac{1}{\lambda}\int_0^\lambda q_X^+(t)dt$

  A coherent measure of risk.

- Weighted VaR: $\rho_\mu(X) := \int_{[0,1]} CVaR_\gamma \mu(d\gamma) = -\int_0^1 q_X(t)\phi(t)dt$

  with μ a probability measure on $[0,1]$ and $\phi(t) := \int_{(t,1]} \frac{1}{s}\mu(ds)$.

  A coherent measure of risk.

Definition 0.1 *A convex measure of risk $\rho : L^p \to \mathbb{R} \cup \infty$ is λ-quantile dependent, if*

$$X\mathbb{I}_{\{X \leq q_X^+(\lambda)\}} = Y\mathbb{I}_{\{Y \leq q_Y^+(\lambda)\}} \quad \mathbf{P}- \;\; a.s. \quad implies \quad \rho(X) = \rho(Y),$$

i.e., ρ depends on the the random variables only up to their λ-quantiles.

Reward: weaker Fatou property required for the representation.

Fatou property of a λ-quantile dependent risk measure:

Definition 0.2 *(λ-quantile Fatou property)*

For any sequence $(X_n) \subset L^p$ such that $q^+_{X_n}(\lambda) \leq c_\lambda$, for some $c_\lambda \in \mathbb{R}$ and all $n \in \mathbb{N}$,

$$X_n \to X \ \mathbf{P} - a.s. \text{ for some } X \in L^p \quad \text{implies} \quad \rho(X) \leq \liminf_{n\to\infty} \rho(X_n).$$

Comparison:

Delbaen (2000): $|X_n| \leq C$,

Biagini and Frettelli (2009): $|X_n| \leq Y$.

So: $|X_n| \leq C \Rightarrow |X_n| \leq Y \Rightarrow q^+_{X_n}(\lambda) \leq c_\lambda$.

If only the losses of the financial positions are considered, then 0 will be a natural upper bound of the quantiles.

Robust representation of λ-quantile dependent convex measures of risk:

Theorem 0.3 *Let $\rho : L^p \to \mathbb{R} \cup \infty$, $1 \leq p \leq \infty$, be a proper λ-quantile dependent convex measure of risk, then the following are equivalent:*

1. *ρ is $\sigma(L^p, (L^p)')$-lower semicontinuous.*

2. *$\rho(X) = \sup_{\mathbf{Q} \in \mathcal{Q}_p}(\mathbb{E}_{\mathbf{Q}}[-X] - \rho^*(\mathbf{Q}))$,*

   with ρ^* the Fenchel-Legendre transformation of ρ and

   $\mathcal{Q}_p = \{\mathbf{Q}$ probability measures $: \mathbf{Q} \ll \mathbf{P}, \frac{d\mathbf{Q}}{d\mathbf{P}} \in (L^p)'\}$.

3. *ρ is continuous from above.*

4. *ρ has the λ-quantile Fatou property.*

Sketch of Proof:

"1⇒2⇒3": see Theorem 4.31 of Föllmer and Schied (2004) for L^∞ case or Theorem 3.3 of Kaina and Rüschendorf (2009) for L^p case.

"3⇒ 4": Continuous from above⇒ ρ has Fatou property (BF(2009))

⇒ ρ has λ-quantile dependent Fatou property.

"4⇒1": Show that $\mathcal{C} := \{\rho < c\}$ is weakly closed. Equivalently, show $\mathcal{C}_r := \mathcal{C} \cap \{X \in L^p : \|X\|_p \leq r\}$ is weakly closed for all $r > 0$.

For $(X_n) \subset \mathcal{C}_r$ s.t. $X_n \to X$ in L^p-norm, $\exists$ subsequence X_{n_k} s.t.

$X_{n_k} \to X$ $\mathbf{P}$-a.s.

$\Rightarrow q^+_{X_{n_k}}(\lambda)$ is uniformly bounded$\Rightarrow \rho(X) \leq \liminf_{n\to\infty} \rho(X_n) \leq c$,

$\Rightarrow X \in \mathcal{C}_r$ and $\mathcal{C}_r$ is strongly closed $\Rightarrow \mathcal{C}_r$ is weakly closed.

An Example: The λ-quantile dependent Weighted VaR.

Definition 0.4 $\rho_{\mu,\lambda} : L^p \to \mathbb{R} \cup \infty$, $1 \leq p \leq \infty$, *is defined as*

$$\rho_{\mu,\lambda}(X) = \int_{[0,\lambda]} CVaR_\gamma(X)\mu(d\gamma) = -\int_0^\lambda q_X(t)\phi(t)dt = \int_0^\lambda q_X(t)q_{\nu_\phi}(t)dt.$$

where μ *is a probability measure on* $[0,1]$ *s.t.* $\mu((\lambda,1]) = 0$, *and assume* $\mu(\{0\}) = 0$.

And $-\phi(t) = \int_{(t,\lambda]} \frac{1}{s}\mu(s)$ *is monotone increasing on* $[0,\lambda]$, *it can be viewed as a quantile function of a probability distribution measure* ν_ϕ *defined as* $\nu_\phi([-\phi(0), -\phi(t)]) := t$, *and* $\nu_\phi(0) = 1-\lambda$.

Then $q_{\nu_\phi}(t) := -\phi(t)$, $\forall t \in [0,\lambda]$.

Notice that $\int_0^\lambda q_{\nu_\phi}(t)dt = -1$.

$\rho_{\mu,\lambda}$ is λ-quantile law invariant:

Definition 0.5 *A convex risk measure $\rho : L^p \to \mathbb{R} \cup \infty$ is λ-quantile law invariant, if for any $X, Y \in L^p$,*

$X\mathbb{I}_{\{X \le q_X^+(\lambda)\}}$ *and* $Y\mathbb{I}_{\{Y \le q_Y^+(\lambda)\}}$ *have same distribution implies* $\rho(X) = \rho(Y)$.

Recall the definition of λ-quantile dependent convex measure of risk:

ρ is λ-quantile dependent, if for any $X, Y \in L^p$,

$$X\mathbb{I}_{\{X \le q_X^+(\lambda)\}} = Y\mathbb{I}_{\{Y \le q_Y^+(\lambda)\}} \quad \mathbf{P}- \text{ a.s. } \quad \text{implies} \quad \rho(X) = \rho(Y),$$

λ-quantile uniform preference (second order stochastic dominance) of two probability distribution measures μ and ν:

Definition 0.6 *Let μ, ν be two probability distribution measures. μ is λ-quantile uniformly preferred over ν if for any "λ-quantile utility function" u defined as $u(x) = u_0(x)\mathbb{I}_{\{x \le q_\nu(\lambda)\}} + u_0(q_\nu(\lambda))\mathbb{I}_{\{x > q_\nu(\lambda)\}}$ with u_0 a utility function, the following is true:*

$$\int_0^\lambda u\, d\mu \ge \int_0^\lambda u d\nu.$$

Proposition 0.7

$$\mu \underset{uni(\lambda)}{\succcurlyeq} \nu \Longleftrightarrow \int_0^t q_\mu(s)ds \ge \int_0^t q_\nu(s)ds,\ \forall t \in [0,\lambda]$$

$$\Longleftrightarrow \int_0^\lambda h(t)q_\mu(t)dt \ge \int_0^\lambda h(t)q_\nu(t)dt,\ \forall \text{ decreasing } h : [0,\lambda] \to \mathbb{R}^+.$$

Robust representation of $\rho_{\mu,\lambda}$:

Recall

$$\rho_{\mu,\lambda}(X) = \int_{[0,\lambda]} CVaR_\gamma(X)\mu(d\gamma) = -\int_0^\lambda q_X(t)\phi(t)dt = \int_0^\lambda q_X(t)q_{\nu_\phi}(t)dt.$$

Lemma 0.8 *Define*

$$\Phi := \left\{ \nu : \nu \text{ distribution measure such that } \nu \underset{uni(\lambda)}{\succcurlyeq} \nu_\phi \text{ and } \int_0^\lambda q_\nu(t)dt = -1 \right\}.$$

Then for $X \in L^p$,

$$\rho_{\mu,\lambda}(X) = \max_{\nu\in\Phi} \int_0^\lambda q_X(t)q_\nu(t)dt.$$

The maximum is obtained by taking $\tilde{\nu} \in \Phi$ s.t. $q_{\tilde{\nu}} = q_{\nu_\phi}$.

Theorem 0.9 *(Robust representation of $\rho_{\mu,\lambda}$)*

For $X \in L^p$, $1 \leq p \leq \infty$,

$$\rho_{\mu,\lambda}(X) = \max_{\mathbf{Q}\in\mathcal{Q}} \mathbb{E}_{\mathbf{Q}}[-X],$$

with $\mathcal{Q} :=$

$$\left\{\mathbf{Q} \text{ probability measure} : \mathbf{Q} \ll \mathbf{P}\,,\ \nu_{-\frac{d\mathbf{Q}}{d\mathbf{P}}} \underset{uni(\lambda)}{\succcurlyeq} \nu_\phi \ \text{ and } \ \int_0^\lambda q_{\nu_{-\frac{d\mathbf{Q}}{d\mathbf{P}}}}(t)dt = -1\right\}.$$

The maximum is obtained by choosing $\mathbf{Q}_X \in \mathcal{Q}$ such that $\frac{d\mathbf{Q}_X}{d\mathbf{P}} = f(X)$,

where the decreasing function f is given by:

$$f(x) = \begin{cases} \phi(F_X(x)) & \text{if } x \text{ is a continuity point of } F_X, \\ \frac{1}{F_X(x)-F_X(x-)} \int_{F_X(x-)}^{F_X(x)} \phi(t)dt & \text{if } x \text{ is a discrete point of } F_X, \end{cases}$$

for $F_X(x) \leq \lambda$ and $f(x) = 0$, otherwise.

Example: $CVaR_\lambda$ for $\lambda \in (0,1)$.

Take $\mu(ds) = \mathbb{I}_{\{\lambda\}}(ds)$, $CVaR_\lambda(X)$ is a special case of $\rho_{\mu,\lambda}(X)$.

It is well known that $CVaR_\lambda(X) = -\frac{1}{\lambda}\int_0^\lambda q_X(t)dt = \sup_{\mathbf{Q}\in\mathcal{Q}_\lambda} \mathbb{E}_{\mathbf{Q}}[-X]$,

where

$$\mathcal{Q}_\lambda = \left\{ \mathbf{Q}\,\text{probability measure} \;: \mathbf{Q} \ll \mathbf{P}, \quad \frac{d\mathbf{Q}}{d\mathbf{P}} \leq \frac{1}{\lambda} \quad \mathbf{P} - a.s. \right\}.$$

The set $\mathcal{Q}_\lambda$ coincides with the set $\mathcal{Q}$ defined in the Theorem:

$$\mathcal{Q} = \left\{ \mathbf{Q}\,\text{probability measure} \;: \mathbf{Q} \ll \mathbf{P}, \nu_{-\frac{d\mathbf{Q}}{d\mathbf{P}}} \underset{uni(\lambda)}{\succcurlyeq} \nu_\phi, \int_0^\lambda q_{\nu_{-\frac{d\mathbf{Q}}{d\mathbf{P}}}}(t)dt = -1 \right\}.$$

To verify, consider a $\mathbf{Q} \in \mathcal{Q}$. Recall that

$$\rho_{\mu,\lambda}(X) = \int_{[0,\lambda]} CVaR_\gamma(X)\mu(d\gamma) = -\int_0^\lambda q_X(t)\phi(t)dt = \int_0^\lambda q_X(t)q_{\nu_\phi}(t)dt,$$

where $\nu_\phi([-\phi(0), -\phi(t)]) := t$, $q_{\nu_\phi}(t) = -\phi(t)$, $\forall t \in [0,\lambda]$.

In this case, $-\phi(t) = -\int_{(t,1]} \frac{1}{s}\mu(ds) = -\frac{1}{\lambda}\mathbb{I}_{[0,\lambda)}(t)$ and $q_{\nu_\phi}(t) = -\phi(t)$.

$$\nu_{-\frac{d\mathbf{Q}}{d\mathbf{P}}} \underset{uni(\lambda)}{\succcurlyeq} \nu_\phi \Leftrightarrow \int_0^t q_{-\nu_{-\frac{d\mathbf{Q}}{d\mathbf{P}}}}(s)ds \geq \int_0^\lambda q_{\nu_\phi}(t)dt = \int_0^t -\frac{1}{\lambda}\mathbb{I}_{[0,\lambda)}(s)ds.$$

If there is a $s \in [0,\lambda]$ s.t. $\frac{d\mathbf{Q}}{d\mathbf{P}}(s) > \frac{1}{\lambda}$, then $q_{-\nu_{\frac{d\mathbf{Q}}{d\mathbf{P}}}}(s) < -\frac{1}{\lambda}$, and since $q_{-\nu_{\frac{d\mathbf{Q}}{d\mathbf{P}}}}(t)$ is monotone increasing,

$$-1 = \int_0^\lambda q_{-\nu_{\frac{d\mathbf{Q}}{d\mathbf{P}}}}(t)dt = \int_0^s q_{-\nu_{\frac{d\mathbf{Q}}{d\mathbf{P}}}}(t)dt + \int_s^\lambda q_{-\nu_{\frac{d\mathbf{Q}}{d\mathbf{P}}}}(t)dt < -\frac{s}{\lambda} - \frac{\lambda - s}{\lambda} = -1.$$

If $\mathbf{Q} \in \mathcal{Q}_\lambda$. Then $\frac{d\mathbf{Q}}{d\mathbf{P}} \leq \frac{1}{\lambda}$ $\mathbf{P}$-a.s. implies

$$\int_0^t q_{\nu_{-\frac{d\mathbf{Q}}{d\mathbf{P}}}}(s)ds \geq -\frac{t}{\lambda} = \int_0^t -\frac{1}{\lambda}\mathbb{I}_{[0,\lambda)}(s)ds = \int_0^t q_{\nu_\phi}(s)ds,$$

for all $t \in [0, \lambda]$. So the distribution measure of $-\frac{d\mathbf{Q}}{d\mathbf{P}}$ is λ-quantile uniformly preferred over ν_ϕ, and it is also true that

$\int_0^\lambda q_{\nu_{-\frac{d\mathbf{Q}}{d\mathbf{P}}}}(t)dt = \mathbb{E}[-\frac{d\mathbf{Q}}{d\mathbf{P}}] = -1.$

THANK YOU!