Infinite Horizon Optimal Search Problem with Hiring and Firing Options

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Optimal Stopping Problem at Study

Sequentially hire and fire candidates who arrive at random times.

Feature:

- Random arrival times of candidates or opportunities
- Two interwoven sequences of stopping times
- Infinite time horizon

Application:

- Similar to the 'secretary problem' but with hiring and firing options
- Job search problem with accepting and quitting options
- Investment and de-investment problem with repeat opportunities

One Candidate

- Candidate process Y_t is an $\mathbb{R}$-valued Itô diffusion process on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F})_{t\geq 0}, P^{s,\nu})$:

$$dY_t = \mu(Y_t)dt + \sigma(Y_t)dB_t, \quad \forall t \geq s,$$

with initial distribution

$$P^{s,\nu}(Y_s \in F) = \nu(F), \quad \forall F \in \mathcal{B}(\mathbb{R}) \text{ (Borel sigma-algebra on } \mathbb{R}).$$

Note that $P^{s,y}$ is the family of probability measures accompanying the strong Markov family Y_t with initial value y such that

$$P^{s,\nu}(F) = \int_{\mathbb{R}} P^{s,y}(F)\nu(dy), \quad \forall F \in \mathcal{F}.$$

- Choose optimal hiring time τ and firing time ζ:

$$v(y) = \sup_{(\tau,\zeta)\in\mathcal{T}} E^{0,y}\left[\left(\int_\tau^\zeta e^{-rt}f(Y_t)dt + e^{-r\tau}N(Y_\tau) + e^{-r\zeta}K(Y_\zeta)\right)\mathbb{I}_{\{\tau<\zeta\}}\right],$$

where $s = 0$, $Y_0 = y$ and

$$\mathcal{T} = \{(\tau, \zeta) : \tau \text{ and } \zeta \text{ are } P^{0,y}\text{-a.s. finite stopping times such that } \tau \leq \zeta\}.$$

Multiple Candidates

- Raw processes Y_t^i are i.i.d. with dynamics given by

$$dY_t^i = \mu(Y_t^i)dt + \sigma(Y_t^i)dB_t^i, \quad \forall t \geq s, \quad i = 0, 1, 2, ...,$$

and initial distribution ν

$$P^{s,\nu}(Y_s^i \in F) = \nu(F), \quad \forall F \in \mathcal{B}(\mathbb{R}), \quad i = 0, 1, 2, ...$$

- Waiting times $s_1, s_2, ...$ is a sequence of i.i.d. random variables whose moment generating function exist

$$\chi(u) = E[e^{-us_i}] < \infty, \quad i = 1, 2, ...$$

- Decision times are

$$\mathcal{S} = \{ (\tau_i, \zeta_i) : (\tau_i, \zeta_i) \text{ are a.s. finite stopping times such that}$$
$$0 \leq \tau_0 \leq \zeta_0 \leq T_1 \leq \tau_1 \leq \zeta_1 \leq T_2 \leq \tau_2 \leq ... \}, \quad \text{where } T_i = \zeta_{i-1} + s_i.$$

- Candidate processes are

$$Z_t^0 = Y_t^0, \quad (Y_0^0 = y)$$
$$Z_t^i = Y_t^i \circ \theta_{T_i}^{-1}, \quad t \geq T_i, \quad i = 1, 2, ...,$$

where θ_s is the shift operator, i.e.,

$$dZ_t^i = \mu(Z_t^i)dt + \sigma(Z_t^i)dB_t^i, \quad \forall t \geq T_i.$$

Infinite Horizon Optimal Stopping Problem

Main problem:

$$v(y) =$$
$$\sup_{(\tau_i,\zeta_i)\in\mathcal{S}} E^y \left[\sum_{i=0}^{\infty} \left(\int_{\tau_i}^{\zeta_i} e^{-rt} f(Z_t^i) dt + e^{-r\tau_i} N(Z_{\tau_i}^i) + e^{-r\zeta_i} K(Z_{\zeta_i}^i) \right) \mathbb{I}_{\{\tau_i<\zeta_i\}} \right].$$

Viewed as a restarting problem:

$$\begin{aligned} v(y) = \sup_{(\tau,\zeta)\in\mathcal{T}} E^y \Bigg[& \left(\int_{\tau}^{\zeta} e^{-rt} f(Y_t) dt + e^{-r\tau} N(Y_\tau) + e^{-r\zeta} K(Y_\zeta) \right) \mathbb{I}_{\{\tau<\zeta\}} \\ & + e^{-r\zeta} \chi(r) \int_{\mathbb{R}} v(x)\nu(dx) \Bigg]. \end{aligned}$$

Here
$$e^{-r\zeta}\chi(r) \int_{\mathbb{R}} v(x)\nu(dx) = E^y \left[e^{-rT_1} v(Z_{T_1}^1) \right].$$

A Little History

Standard optimal stopping problem:

$$v(y) = \sup_{\zeta} \mathbb{E}^y \left[\int_0^{\zeta} e^{-rt} f(Y_t) dt + e^{-r\zeta} K(Y_\zeta) \right]$$

Joint work with Masahiko Egami (2008):

$$v(y) = \sup_{\zeta \le \tau} \mathbb{E}^y \left[\int_0^{\zeta} e^{-rt} f(Y_t) dt + e^{-r\zeta} K(Y_\zeta) - \int_0^{\tau} c e^{-rt} dt + e^{-r\tau} X_\tau^i \right]$$

Current problem:

$$v(y) = \sup_{\tau \le \zeta} E^y \left[\left(\int_\tau^{\zeta} e^{-rt} f(Y_t) dt + e^{-r\tau} N(Y_\tau) + e^{-r\zeta} K(Y_\zeta) \right) \mathbb{I}_{\{\tau < \zeta\}} \right.$$
$$\left. + e^{-r\zeta} \chi(r) \int_{\mathbb{R}} v(x) \nu(dx) \right]$$

Simple Brownian Model with Linear Benefit/Cost

- Let $\mu(\cdot) = 0$, $\sigma(\cdot) = 1$, then $Y_t = B_t$ is a standard Brownian motion starting at y.
- The benefit and cost functions are assumed to be linear: $f(x) = ax, N(x) = cx, K(x) = bx$.
- Assume that the constants satisfy $a - br > 0, a + cr > 0, b + c > 0$.
- One candidate problem:

$$v(y) = \sup_{(\tau,\zeta)\in\mathcal{T}} E^{0,y}\left[\left(\int_\tau^\zeta e^{-rt} aB_t dt + e^{-r\tau} cB_\tau + e^{-r\zeta} bB_\zeta\right)\mathbb{I}_{\{\tau<\zeta\}}\right].$$

- Associate to a pair of Markov decision problems sequentially,

$$u(y) = \sup_\zeta E^{0,y}\left[\int_0^\zeta e^{-rt} aB_t dt + e^{-r\zeta} bB_\zeta\right],$$
$$v(y) = \sup_\tau E^{0,y}\left[\left(e^{-r\tau} cB_\tau + e^{-r\tau} u(B_\tau)\right)\mathbb{I}_{\{\zeta^*>0\}}\right],$$

where ζ^* is the optimal stopping time for achieving value function $u(y)$.

Solution to One Candidate Problem

The solution to the value function

$$u(y) = \sup_{\zeta} E^{0,y}\left[\int_0^{\zeta} e^{-rt} a B_t dt + e^{-r\zeta} b B_{\zeta}\right]$$

is standard to compute. When $a - br > 0$,

$$u(y) = \begin{cases} \frac{a}{r} y + \left(b - \frac{a}{r}\right) L^* e^{-(y-L^*)\sqrt{2r}}, & \text{for } y > L^*; \\ by, & \text{for } y \le L^*, \end{cases}$$

where $L^* = -\frac{1}{\sqrt{2r}}$. The continuation region is

$$\mathcal{C}_u = (L^*, \infty)$$

and the optimal stopping time is the first exit time of the continuation region

$$\zeta^* = \inf\{t : B_t \notin \mathcal{C}_u\} = \inf\{t : B_t \le L^*\}.$$

Solution to One Candidate Problem Continued

The solution to the value function

$$v(y) = \sup_{\tau} E^{0,y}\left[\left(e^{-r\tau} c B_\tau + e^{-r\tau} u(B_\tau)\right) \mathbb{I}_{\{\zeta^* > 0\}}\right],$$

where

$$u(y) = \begin{cases} \frac{a}{r} y + \left(b - \frac{a}{r}\right) L^* e^{-(y-L^*)\sqrt{2r}}, & \text{for } y > L^*; \\ by, & \text{for } y \le L^*, \end{cases}$$

is

$$v(y) = \begin{cases} \left(c + \frac{a}{r}\right) y + \left(b - \frac{a}{r}\right) L^* e^{-(y-L^*)\sqrt{2r}}, & \text{for } y \ge U^*; \\ \left(\left(c + \frac{a}{r}\right) U^* + \left(b - \frac{a}{r}\right) L^* e^{-(U^*-L^*)\sqrt{2r}}\right) e^{-(U^*-y)\sqrt{2r}}, & \text{for } y < U^*, \end{cases}$$

where U^* is the solution to

$$2\left(b - \frac{a}{r}\right) e^{-(U-L^*)\sqrt{2r}} = \left(c + \frac{a}{r}\right)(\sqrt{2r}U - 1),$$

when $a + cr > 0, b + c > 0$. The pair of optimal stopping time is thus $(\tau^*, \zeta^* \circ \theta_{\tau^*})$, where

$$\tau^* = \inf\{t : B_t \ge U^*\}, \quad \zeta^* \circ \theta_{\tau^*} = \inf\{t \ge \tau^* : B_t \le L^*\}.$$

Solution to Multiple Candidate Problem

- Multiple candidate problem:

$$v(y) = \sup_{(\tau,\zeta)\in\mathcal{T}} E^y \left[\left(\int_\tau^\zeta e^{-rt} aB_t dt + e^{-r\tau} cB_\tau + e^{-r\zeta} bB_\zeta \right) \mathbb{I}_{\{\tau<\zeta\}} \right.$$
$$\left. + e^{-r\zeta}\chi(r) \int_{\mathbb{R}} v(x)\nu(dx) \right].$$

- Associate to a pair of Markov decision problems sequentially,

$$u(y) = \sup_\zeta E^{0,y} \left[\int_0^\zeta e^{-rt} aB_t dt + e^{-r\zeta} bB_\zeta + e^{-r\zeta}\chi(r) \int_{\mathbb{R}} v(x)\nu(dx) \right],$$
$$v(y) = \sup_\tau E^{0,y} \left[\left(e^{-r\tau} cB_\tau + e^{-r\tau} u(B_\tau) \right) \mathbb{I}_{\{\zeta^*>0\}} \right.$$
$$\left. + \left(e^{-r\tau}\chi(r) \int_{\mathbb{R}} v(x)\nu(dx) \right) \mathbb{I}_{\{\zeta^*=0\}} \right].$$

Solution to Multiple Candidate Problem

If there exist unique solutions (L^*, U^*, m^{L^*,U^*}) to equations

$$
\begin{aligned}
&m^{L,U} \sinh((U-L)\sqrt{2r}) = \gamma - \beta e^{-(U-I)\sqrt{2r}},\\
&m^{L,U} \cosh((U-L)\sqrt{2r}) = \gamma\sqrt{2r}U + \beta e^{-(U-I)\sqrt{2r}},\\
&m^{L,U}\left[\frac{1}{\chi(r)} - \Phi(L) - M^-(L,U) - \cosh((U-L)\sqrt{2r})M^+(L,U)\right]\\
&\qquad\qquad\qquad\qquad = \gamma\sqrt{2r}\Psi(U) + \beta e^{I\sqrt{2r}}\Gamma(U),
\end{aligned}
$$

which satisfy $L^* \leq I^* = \frac{m^{L^*,U^*}}{\beta\sqrt{2r}} - \frac{1}{\sqrt{2r}} < U^*$, where

$$
\beta = \frac{1}{\sqrt{2r}}\left(\frac{a}{r} - b\right), \quad \gamma = \frac{1}{\sqrt{2r}}\left(\frac{a}{r} + c\right),
$$

$$
\Phi(L) = \int_{(-\infty,L]} \nu(dx), \quad \Psi(U) = \int_{[U,\infty)} x\nu(dx), \quad \Gamma(U) = \int_{[U,\infty)} e^{-x\sqrt{2r}}\nu(dx),
$$

$$
M^-(L,U) = \frac{\int_{(L,U)} \sinh\big((U-x)\sqrt{2r}\big)\nu(dx)}{\sinh\big((U-L)\sqrt{2r}\big)}, \qquad M^+(L,U) = \frac{\int_{(L,U)} \sinh\big((x-L)\sqrt{2r}\big)\nu(dx)}{\sinh\big((U-L)\sqrt{2r}\big)},
$$

Solution to Multiple Candidate Problem Continued

then the value functions are

$$u^{I^*}(y) = \begin{cases} \frac{a}{r} y + \beta e^{-(y - I^*)\sqrt{2r}}, & \text{for } y > I^*; \\ by + m^{L^*,U^*}, & \text{for } y \le I^*, \end{cases}$$

$$v^{L^*,U^*}(y) = \begin{cases} m^{L^*,U^*}, & \text{for } y \le L^*; \\ \left[cU^* + u^{I^*}(U^*)\right] \frac{\sinh((y-L^*)\sqrt{2r})}{\sinh((U^*-L^*)\sqrt{2r})} \\ \qquad + m^{L^*,U^*} \frac{\sinh((U^*-y)\sqrt{2r})}{\sinh((U^*-L^*)\sqrt{2r})}, & \text{for } L^* < y < U^*; \\ \left(\frac{a}{r} + c\right) y + \beta e^{-(y-I^*)\sqrt{2r}}, & \text{for } y \ge U^*, \end{cases}$$

and the Markov decision are made through

- continuation region: $\mathcal{C}_v = (L^*, U^*)$,
- hiring region: $\mathcal{H}_v = [U^*, \infty)$,
- firing region: $\mathcal{F}_v = (-\infty, L^*]$,
- continuation region: $\mathcal{C}_u = (I^*, \infty)$,
- firing region: $\mathcal{F}_u = (-\infty, I^*]$.

General Optimality Theorem

Theorem 1 (Variational Inequality) *Suppose there exist adapted and continuous stochastic processes* $(U_t^{s,y})_{t\geq s}$ *and* $(V_t^{s,y})_{t\geq s}$ *for which the following conditions hold for all* $y\in\mathbb{R}$*:*

a. $e^{-rt}U_t^{s,y} \geq \int_s^t e^{-ru} f(Y_u)du + e^{-rt}K(Y_t) + e^{-rt}\chi(r)\int_{\mathbb{R}} V_0^{0,z}\nu(dz)$*,* $\forall t\geq s,\ P^{s,y}-a.s.$*,*

b. $e^{-rt}U_t^{s,y}$ *is a uniformly integrable supermartingale, and*

c. *there exists a stopping times* $\zeta^*\in\mathcal{R}_s$ *such that*

$$U_s^{s,y} = E^{s,y}\left[\int_s^{\zeta^*} e^{-rt}f(Y_t)dt + e^{-r\zeta^*}K(Y_{\zeta^*}) + e^{-r\zeta^*}\chi(r)\int_{\mathbb{R}} V_0^{0,z}\nu(dz)\right];$$

d. $e^{-rt}V_t^{0,y} \geq e^{-rt}N(Y_t) + e^{-rt}U_t^{t,Y_t}$*,* $\forall t\geq 0,\ P^{0,y}-a.s.$

e. $V_t^{0,y} \geq \chi(r)\int_{\mathbb{R}} V_0^{0,z}\nu(dz)$*,* $\forall t\geq 0,\ P^{0,y}-a.s.$

f. $e^{-rt}V_t^{0,y}$ *is a uniformly integrable supermartingale, and*

g. *there exists a stopping times* $\tau^*\in\mathcal{R}_0$ *such that*

$$V_0^{0,y} = E^{0,y}\left[\left(e^{-r\tau^*}N(Y_{\tau^*}) + e^{-r\tau^*}U_{\tau^*}^{\tau^*,Y_{\tau^*}}\right)\mathbb{I}_{\{\tau^*<\zeta^*\}} + \left(e^{-r\tau^*}\chi(r)\int_{\mathbb{R}} V_0^{0,z}\nu(dz)\right)\mathbb{I}_{\{\tau^*=\zeta^*\}}\right],$$

where $\zeta^*\in\mathcal{R}_{\tau^*}$ *is the optimal stopping time for achieving* $U_{\tau^*}^{\tau^*,Y_{\tau^*}}$ *in condition* ***(c)****. If* $N(y)+U_0^{0,y}\leq\chi(r)\int_{\mathbb{R}} V_0^{0,z}\nu(dz)$ *for all* $y\in\mathbb{R}$*, then let* $\tau^*=\zeta^*=0$ *and* $v(y)=0$ *is the optimal value function. Otherwise* $V_0^{0,y}$ *is the optimal value function, and* (τ^*,ζ^*) *is a pair of optimal stopping times.*

Corollary 2 (Least Superharmonic Majorant) *Suppose that $u : \mathbb{R} \to \mathbb{R}$ and $v : \mathbb{R} \to \mathbb{R}$ are continuous functions that jointly satisfy the conditions*

h. $u(x) \geq K(x) + \chi(r) \int_{\mathbb{R}} v(z)\nu(dz)$, $\forall\, x \in \mathbb{R}$,

i. $e^{-rt}u(Y_t) + \int_0^t e^{-ru} f(Y_u)du$ *is a* $P^{0,y}$ *uniformly integrable supermartingale for all* $y \in \mathbb{R}$, *and*

j. *for each* $y \in \mathbb{R}$ *there exists a stopping time* $\zeta^* \in \mathcal{R}_0$ *such that*

$$u(y) = E^{0,y}\left[\int_0^{\zeta^*} e^{-ru} f(Y_u)du + e^{-r\zeta^*}K(Y_{\zeta^*}) + e^{-r\zeta^*}\chi(r)\int_{\mathbb{R}} v(x)\nu(dx)\right];$$

k. $v(x) \geq \max\left\{u(x) + N(x),\ \chi(r)\int_{\mathbb{R}} v(z)\nu(dz)\right\}$, $\forall\, x \in \mathbb{R}$,

l. $e^{-rt}v(Y_t)$ *is a* $P^{0,y}$ *uniformly integrable supermartingale for all* $y \in \mathbb{R}$, *and*

m. *for each* $y \in \mathbb{R}$ *there exists a stopping time* $\tau^* \in \mathcal{R}_0$ *such that*

$$v(y) = E^{0,y}\left[\left(e^{-r\tau^*}N(Y_{\tau^*}) + e^{-r\tau^*}u(Y_{\tau^*})\right)\mathbb{I}_{\{\zeta^*>0\}} + \left(e^{-r\tau^*}\chi(r)\int_{\mathbb{R}} v(x)\nu(dx)\right)\mathbb{I}_{\{\zeta^*=0\}}\right],$$

where $\zeta^* \in \mathcal{T}_{\tau^*}$ *is the solution to (**j**) for* $u(Y_{\tau^*})$, *i.e.,*

$$u(Y_{\tau^*}) = E^{0,Y_{\tau^*}}\left[\int_0^{\zeta^*} e^{-ru} f(Y_u)du + e^{-r\zeta^*}K(Y_{\zeta^*}) + e^{-r\zeta^*}\chi(r)\int_{\mathbb{R}} v(x)\nu(dx)\right].$$

If $u(x) + N(x) \leq \chi(r)\int_{\mathbb{R}} v(z)\nu(dz)$ *for all* $x \in \mathbb{R}$, *then let* $\tau^* = \zeta^* = 0$ *and* $v(y) = 0$ *is the optimal value function. Otherwise, the optimal value function* $v(y)$ *is a positive function, and* $(\tau^*, \zeta^* \circ \theta_{\tau^*})$ *is a pair of optimal stopping times.*

Corollary 3 (Variational Inequality with Smooth Pasting) *Suppose* $u : \mathbb{R} \to \mathbb{R}$ *and* $v : \mathbb{R} \to \mathbb{R}$ *satisfies*

1. $u \in C^1(\mathbb{R})$ *and* $v \in C^1(\mathbb{R})$,
2. $u(x) \geq K(x) + \chi(r) \int_{\mathbb{R}} v(z)\nu(dz)$, $\forall\, x \in \mathbb{R}$,
   Now define the continuation region as $\mathcal{C}_u = \{x \in \mathbb{R} \,:\, u(x) > K(x) + \chi(r) \int_{\mathbb{R}} v(z)\nu(dz)\}$ *and the firing region as* $\mathcal{F}_u = \{x \in \mathbb{R} \,:\, u(x) = K(x) + \chi(r) \int_{\mathbb{R}} v(z)\nu(dz)\}$,
3. $v(x) \geq \max\left\{u(x) + N(x),\, \chi(r) \int_{\mathbb{R}} v(z)\nu(dz)\right\}$, $\forall x \in \mathbb{R}$,
   Now define the continuation region as
   $\mathcal{C}_v = \left\{x \in \mathbb{R} \,:\, v(x) > \max\left\{u(x) + N(x),\, \chi(r) \int_{\mathbb{R}} v(z)\nu(dz)\right\}\right\}$,
   the hiring region as $\mathcal{H}_v = \{x \in \mathbb{R} \,:\, v(x) = u(x) + N(x)\}$,
   and the firing region as $\mathcal{F}_v = \left\{x \in \mathbb{R} \,:\, v(x) = \chi(r) \int_{\mathbb{R}} v(z)\nu(dz)\right\}$,
4. Y_t *spends zero local time on* $\partial\mathcal{C}_u \cup \partial\mathcal{C}_v$ *a.s.:* $E^{0,y}[\int_0^\infty \mathbb{I}_{\partial\mathcal{C}_u}(Y_t)dt] = 0$ *and* $E^{0,y}[\int_0^\infty \mathbb{I}_{\partial\mathcal{C}_v}(Y_t)dt] = 0$, $\forall y \in \mathbb{R}$,
5. $\partial\mathcal{C}_u \cup \partial\mathcal{C}_v$ *is a Lipschitz surface,*
6. $u \in C^2(\mathbb{R}\backslash\partial\mathcal{C}_u)$ *and the second order derivative of* u *is locally bounded near* $\partial\mathcal{C}_u$, *and* $v \in C^2(\mathbb{R}\backslash\partial\mathcal{C}_v)$ *and the second order derivative of* v *is locally bounded near* $\partial\mathcal{C}_v$,
7. $Lu + f \leq 0$ *on* $\mathcal{F}_u$, *and* $Lu + f = 0$ *on* $\mathcal{C}_u$, $Lv \leq 0$ *on* $\mathcal{H}_v \cup \mathcal{F}_v$, *and* $Lv = 0$ *on* $\mathcal{C}_v$,
   Now define $\zeta^* = \inf\{t \geq 0 : Y_t \notin \mathcal{C}_u\}$ *and* $\tau^* = \inf\{t \geq 0 : Y_t \notin \mathcal{C}_v\}$,
8. $\zeta^* < \infty$ *and* $\tau^* < \infty$, $P^{0,y}-$*a.s.,*
9. *the family* $\{u(Y_\zeta) : \zeta$ *is a stopping time such that* $\zeta \leq \zeta^*\}$ *is uniformly integrable, and the family* $\{v(Y_\tau) : \tau$ *is a stopping time such that* $\tau \leq \tau^*\}$ *is uniformly integrable,* $\forall y \in \mathbb{R}$,

If $u(x) + N(x) \leq \chi(r) \int_{\mathbb{R}} v(z)\nu(dz)$ *for all* $x \in \mathbb{R}$, *then let* $\tau^* = \zeta^* = 0$ *and* $v(y) = 0$ *is the optimal value function. Otherwise, the optimal value function* $v(y)$ *is a positive function, and* $(\tau^*, \zeta^* \circ \theta_{\tau^*})$ *is a pair of optimal stopping times.*

Future Work

- New candidates arrive at random times $s_1, s_1 + s_2, \dots$ where s_i are i.i.d. random variables. The arrival of the new candidate can immediately affect the decision about the current candidate.

- Limit the model to finite number of candidates.

- Impose an exponentially distributed final time for all operations.

- A random number of candidates arrives simultaneously, but only one candidate will be under consideration.

- At any given time, up to a fixed number of candidates can be under consideration.