A new approach for investment performance measurement

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Performance measurement of investment strategies

Market environment

Riskless and risky securities

- $(\Omega, \mathcal{F}, \mathbb{P})$; $W = (W^1, \ldots, W^d)$ standard Brownian Motion

- Traded securities

$$1 \leq i \leq k \qquad \begin{cases} dS_t^i = S_t^i\Big(\mu_t^i dt + \sigma_t^i \cdot dW_t\Big)\ , & S_0^i > 0 \\ dB_t = r_t B_t dt\ , & B_0 = 1 \end{cases}$$

$\mu_t, r_t \in \mathbb{R}$, $\sigma_t^i \in \mathbb{R}^d$ bounded and $\mathcal{F}_t$-measurable stochastic processes

- Postulate existence of an $\mathcal{F}_t$-measurable stochastic process $\lambda_t \in \mathbb{R}^d$ satisfying

$$\mu_t - r_t\, \mathbb{1} = \sigma_t^T \lambda_t$$

- No assumptions on market completeness

Market environment

- Self-financing investment strategies π_t^0, $\pi_t = (\pi_t^1, \ldots, \pi_t^i, \ldots, \pi_t^k)$

- Present value of this allocation

$$X_t = \sum_{i=0}^{k} \pi_t^i$$

$$dX_t = \sum_{i=1}^{k} \pi_t^i \sigma_t^i \cdot (\lambda_t \, dt + dW_t)$$

$$= \sigma_t \pi_t \cdot (\lambda_t \, dt + dW_t)$$

Traditional framework

A (deterministic) utility datum $u_T(x)$ is assigned at the end of a fixed investment horizon

$$U_T(x) = u_T(x)$$

No market input to the choice of terminal utility

Backwards in time generation of the indirect utility

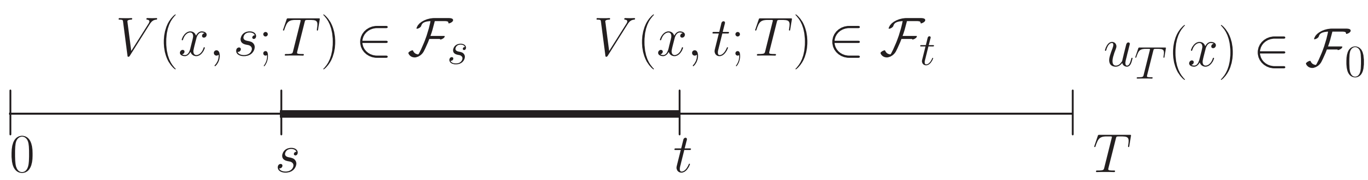
$$V(x, s; T) = \sup_\pi E_\mathbb{P}(u_T(X_T^\pi) | \mathcal{F}_s; X_s^\pi = x)$$

$$V(x, s; T) = \sup_\pi E_\mathbb{P}(V(X_t^\pi, t; T) | \mathcal{F}_s; X_s^\pi = x) \qquad \text{(DPP)}$$

$$V(x, s; T) = E_\mathbb{P}(V(X_t^{\pi^*}, t; T) | \mathcal{F}_s; X_s^{\pi^*} = x)$$

The value function process becomes the intermediate utility for all $t \in [0, T)$

The value function process



- For each self-financing strategy, represented by π, the associated wealth X_t^π satisfies

$$E_{\mathbb{P}}(V(X_t^\pi, t; T)|\mathcal{F}_s) \leq V(X_s^\pi, s; T) \ , \qquad 0 \leq s \leq t \leq T$$

- There exists a self-financing strategy, represented by π^*, for which the associated wealth $X_t^{\pi^*}$ satisfies

$$E_{\mathbb{P}}(V(X_t^{\pi^*}, t; T)|\mathcal{F}_s) = V(X_s^{\pi^*}, s; T) \ , \qquad 0 \leq s \leq t \leq T$$

- At expiration, $\quad V(x,T;T) = u_T(x) \in \mathcal{F}_0$

Study of the value function process

- "Arbitrary" environments

  Duality methods

  Martingale representation results

- Markovian environments

  HJB equation

  Feedback optimal controls

  Weak solutions

  ⋮

A stochastic PDE for the value function process

Intuition

- Assume that, for $t \in [0,T]$, the value function $V(x,t)$ solves

$$dV\left(x,t\right) = b\left(x,t\right)dt + a\left(x,t\right)\cdot dW_t$$

where b, a are $\mathcal{F}_t-$measurable processes.

- Recall that for an arbitrary admissible portfolio π, the associated wealth process, X^{π}, solves

$$dX_t^{\pi} = \sigma_t \pi_t \left(\lambda_t dt + dW_t\right)$$

- Applying the Ito-Ventzell formula to $V\left(X_t^{\pi}, t\right)$ yields

$$dV\left(X_t^{\pi},t\right) = b\left(X_t^{\pi},t\right)dt + a\left(X_t^{\pi},t\right)\cdot dW_t$$

$$+V_x\left(X_t^{\pi},t\right)dX_t^{\pi} + \frac{1}{2}V_{xx}\left(X_t^{\pi},t\right)d\left\langle X^{\pi}\right\rangle_t + a_x\left(X_t^{\pi},t\right)\cdot d\left\langle W, X^{\pi}\right\rangle_t$$

$$= \left(b\left(X_t^{\pi},t\right) + V_x\left(X_t^{\pi},t\right)\sigma_t\pi_t\cdot\lambda_t + \sigma_t\pi_t\cdot a_x(X_t^{\pi},t) + \frac{1}{2}V_{xx}\left(X_t^{\pi},t\right)\left|\sigma_t\pi_t\right|^2\right)dt$$

$$+\left(a\left(X_t^{\pi},t\right) + V_x\left(X_t^{\pi},t\right)\sigma_t\pi_t\right)\cdot dW_t$$

Intuition (continued)

- By the monotonicity and concavity assumptions, the quantity

$$\sup_{\pi}\left(V_x\left(X_t^{\pi},t\right)\sigma_t\pi_t\cdot\lambda_t+\sigma_t\pi_t\cdot a_x(X_t^{\pi},t)+\frac{1}{2}V_{xx}\left(X_t^{\pi},t\right)\left|\sigma_t\pi_t\right|^2\right)$$

is well defined

- Calculating the optimum π^* yields

$$\pi_t^*=-\sigma_t^+\frac{V_x\left(X_t^{\pi^*},t\right)\lambda_t+a_x\left(X_t^{\pi^*},t\right)}{V_{xx}\left(X_t^{\pi^*},t\right)}$$

- Deduce that the above supremum is given by

$$M^*\left(X_t^{\pi^*},t\right)=-\frac{\left|\sigma_t\sigma_t^+\left(V_x\left(X_t^{\pi^*},t\right)\lambda_t+a_x\left(X_t^{\pi^*},t\right)\right)\right|^2}{2V_{xx}\left(X_t^{\pi^*},t\right)}$$

- The drift coefficient b must satisfy

$$b\left(X_t^{\pi^*},t\right)=-M^*\left(X_t^{\pi^*},t\right)$$

SPDE for the value function process

- Market $(\sigma_t, \sigma_t^+, \lambda_t)$; volatility $a(x,t) \in \mathcal{F}_t$

$$dV = \frac{1}{2} \frac{|\sigma\sigma^+ \mathcal{A}(V\lambda + a)|^2}{\mathcal{A}^2 V} dt + a \cdot dW$$

$$V(x,T) = u_T(x) \in \mathcal{F}_0 \; ; \qquad \mathcal{A} = \frac{\partial}{\partial x}$$

- Feedback optimal portfolio vector

$$\pi_t^* = \pi^*(X_t^{\pi,*}, t) = -\sigma^+ \frac{\mathcal{A}(V\lambda + a)}{\mathcal{A}^2 V}(X_t^{\pi,*} t)$$

- Choices for the volatility process a?

A Markovian example

- $r_t = r(Y_t), \quad \mu_t = \mu(Y_t), \quad \sigma_t = \sigma(Y_t)$

  $dY_t = \theta(Y_t)\,dt + \Theta^T(Y_t)\,dW_t$

- Value function

$$v(x, y, t; T) = \sup_{\pi} E\left(u_T(X_T^{\pi}) \mid X_t^{\pi} = x,\ Y_t = y\right)$$

- HJB equation

$$v_t + \sup_{\pi}\left(\frac{1}{2}|\sigma\pi|^2 v_{xx} + \sigma\pi \cdot \sigma\sigma^{+}(\lambda v_x + \Theta v_{xy}) + \frac{1}{2}\Theta^T\Theta \cdot v_{yy} + \theta \cdot v_y\right)$$

$$= v_t - \frac{1}{2}\,\frac{|\sigma\sigma^{+}(v_x\lambda + \Theta v_{xy})|^2}{v_{xx}} + \frac{1}{2}\Theta^T\Theta \cdot v_{yy} + \theta \cdot v_y = 0$$

- The SPDE for the value function process

$$V(x,t) = v(x, Y_t, t; T)$$

$$dV(x,t) = v_t\, dt + v_y \cdot dY + \frac{1}{2} v_{yy} \cdot d\langle Y \rangle$$

$$\overset{\text{HJB}}{=} \left(\frac{1}{2} \frac{|\sigma\sigma^{+}(v_x \lambda + \Theta v_{xy})|^2}{v_{xx}} - \frac{1}{2}\, \Theta^T \Theta \cdot v_{yy} - \theta \cdot v_y \right) dt$$

$$+ v_y \cdot \left(\theta\, dt + \Theta^T\, dW \right) + \frac{1}{2}\, v_{yy} \cdot \Theta^T \Theta\, dt$$

$$= \frac{1}{2} \frac{|\sigma\sigma^{+}(V_x \lambda + a_x(x,t))|^2}{V_{xx}}\, dt + a(x,t) \cdot dW$$

- The volatility process is uniquely determined: $a(x,t) = \Theta v_y(x, Y_t, t; T)$

Going beyond the deterministic terminal utility problem

Motivation (partial)

- Terminal utility might be ω-dependent

  Liability management, indifference valuation

$$u_T(x,\omega) = -e^{-\gamma(x - C_T(\omega))} \; ; \qquad C_T \in \mathcal{F}_T$$

  Numeraire consistency

$$u_T(x,\omega) = -e^{-\gamma_T(\omega)x}$$

- Need to extend the value function process beyond T

- Need to manage liabilities of arbitrary maturities

How do we formulate investment performance criteria?

Investment performance process

$$U(x,t) \text{ is an } \mathcal{F}_t\text{-adapted process}, \qquad t \geq 0$$

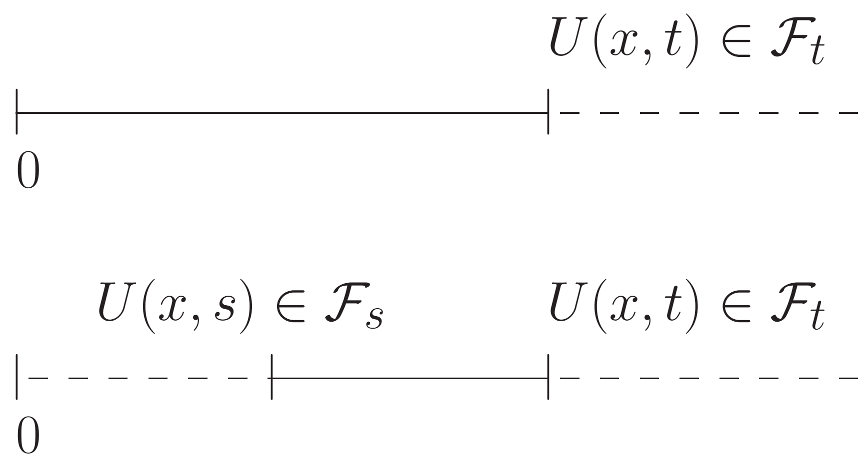
- The mapping $x \to U(x,t)$ is increasing and concave
- For each self-financing strategy, represented by π, the associated (discounted) wealth X_t^{π} satisfies

$$E_{\mathbb{P}}(U(X_t^{\pi},t) \mid \mathcal{F}_s) \leq U(X_s^{\pi},s), \qquad 0 \leq s \leq t$$

- There exists a self-financing strategy, represented by π^*, for which the associated (discounted) wealth $X_t^{\pi^*}$ satisfies

$$E_{\mathbb{P}}(U(X_t^{\pi^*},t) \mid \mathcal{F}_s) = U(X_s^{\pi^*},s), \qquad 0 \leq s \leq t$$

Optimality across times



$$U(x,s) = \sup_{\mathcal{A}} E(U(X_t^{\pi}, t) | \mathcal{F}_s,\ X_s = x)$$

- What is the meaning of this process?
- Does such a process aways exist?
- Is it unique?

Forward performance process

A datum $u_0(x)$ is assigned at the beginning of

the trading horizon, $t = 0$

$$U(x, 0) = u_0(x)$$

Forward in time criteria

$$E_{\mathbb{P}}(U(X_t^{\pi}, t)|\mathcal{F}_s) \leq U(X_s^{\pi}, s), \qquad 0 \leq s \leq t$$

$$E_{\mathbb{P}}(U(X_t^{\pi^*}, t)|\mathcal{F}_s) = U(X_s^{\pi^*}, s), \qquad 0 \leq s \leq t$$

Many difficulties due to "inverse in time"

nature of the problem

The forward performance SPDE

The forward performance SPDE

Let $U(x,t)$ be an $\mathcal{F}_t-$measurable process such that the mapping $x \rightarrow U(x,t)$ is increasing and concave. Let also $U = U(x,t)$ be the solution of the stochastic partial differential equation

$$dU = \frac{1}{2}\frac{\left|\sigma\sigma^{+}\mathcal{A}\left(U\lambda + a\right)\right|^{2}}{\mathcal{A}^{2}U}dt + a \cdot dW$$

where $a = a(x,t)$ is an $\mathcal{F}_t-$adapted process, while $\mathcal{A} =\frac{\partial}{\partial x}$.

Then $U(x,t)$ is a forward performance process.

The process a may depend on t, x, U, its spatial derivatives etc.

Optimal portfolios and wealth

At the optimum

- The optimal portfolio vector π^* is given in the feedback form

$$\pi_t^* = \pi^*\left(X_t^*, t\right) = -\sigma^+ \frac{\mathcal{A}\left(U\lambda + a\right)}{\mathcal{A}^2 U}\left(X_t^*, t\right)$$

- The optimal wealth process X^* solves

$$dX_t^* = -\sigma\sigma^+ \frac{\mathcal{A}\left(U\lambda + a\right)}{\mathcal{A}^2 U}\left(X_t^*, t\right)\left(\lambda dt + dW_t\right)$$

Solutions to the forward performance SPDE

$$dU = \frac{1}{2} \frac{\left| \sigma \sigma^{+} \mathcal{A} \left(U \lambda + a \right) \right|^2}{\mathcal{A}^2 U} dt + a \cdot dW$$

Local differential coefficients

$$a\left(x,t\right) = F\left(x,t,U\left(x,t\right),U_x\left(x,t\right)\right)$$

Difficulties

- The equation is fully nonlinear
- The diffusion coefficient depends, in general, on U_x and U_{xx}
- The equation is not (degenerate) elliptic

The zero volatility case: $a(x,t) \equiv 0$

Space-time monotone performance process

The forward performance SPDE simplifies to

$$dU = \frac{1}{2}\frac{\left|\sigma\sigma^{+}\mathcal{A}\left(U\lambda\right)\right|^{2}}{\mathcal{A}^{2}U}dt$$

The process

$$U\left(x,t\right) = u\left(x,A_t\right) \quad \text{with} \quad A_t = \int_0^t \left|\sigma_s\sigma_s^{+}\lambda_s\right|^2 ds$$

with $u : \mathbb{R} \times [0,+\infty) \rightarrow \mathbb{R}$, increasing and concave with respect to x, and solving

$$u_t u_{xx} = \frac{1}{2}u_x^2$$

is a solution.

MZ (2006)

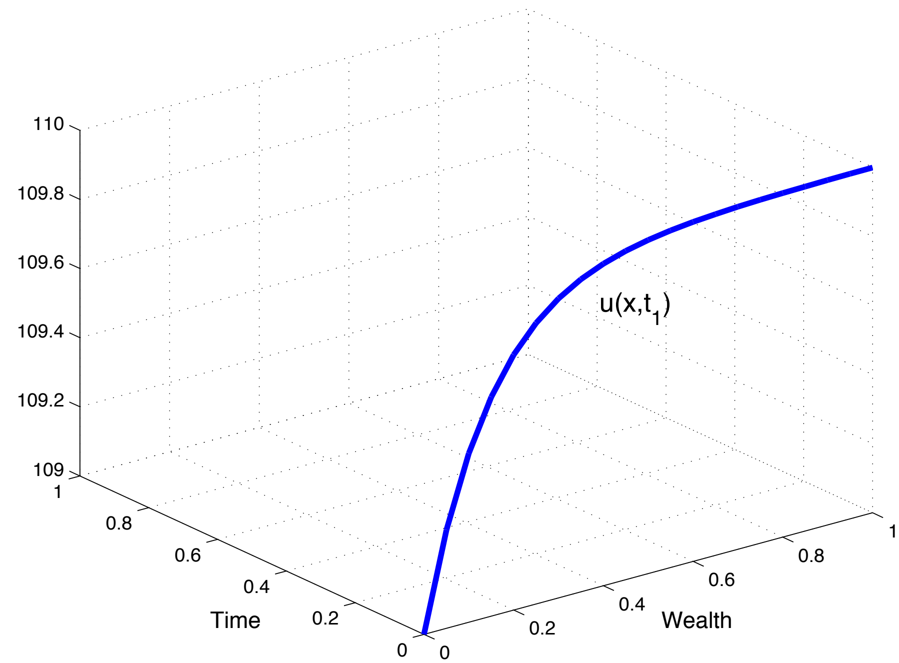
Berrier, Rogers and Tehranchi (2007)

Performance measurement

time t_1, information $\mathcal{F}_{t_1}$

risk premium

$$A_{t_1} = \int_0^{t_1} |\lambda|^2 \, ds \longrightarrow$$



$$A_{t_1} \quad \dashrightarrow \quad + \quad \dashleftarrow \quad u(x,t_1)$$

$$\Downarrow$$

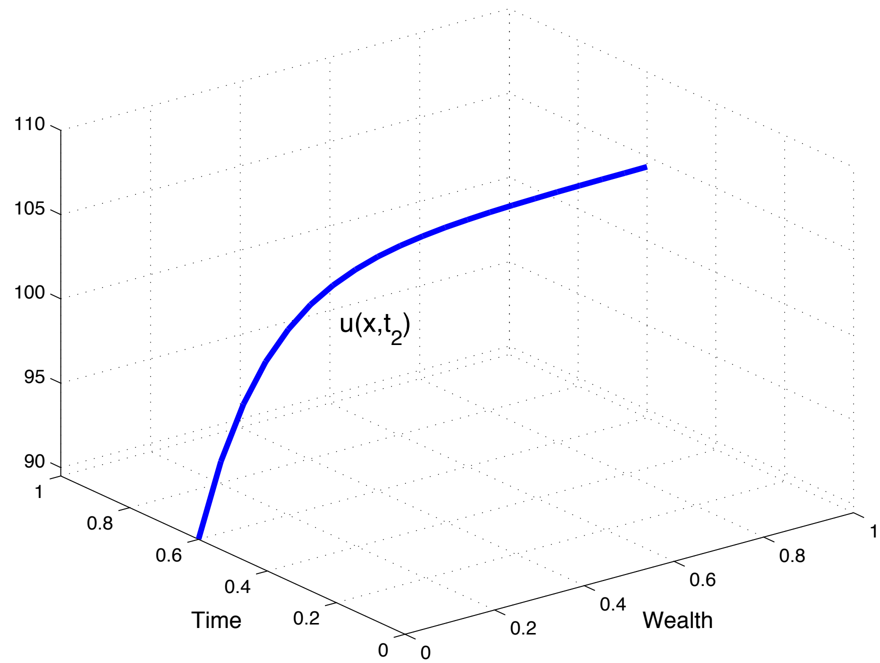
$$\boxed{U(x,t_1) = u(x, A_{t_1}) \in \mathcal{F}_{t_1}}$$

Performance measurement

time t_2, information $\mathcal{F}_{t_2}$

risk premium

$$A_{t_2} = \int_0^{t_2} |\lambda|^2 \, ds \quad \longrightarrow$$



$$A_{t_2} \dashrightarrow + \dashleftarrow u(x, t_2)$$

$$\Downarrow$$

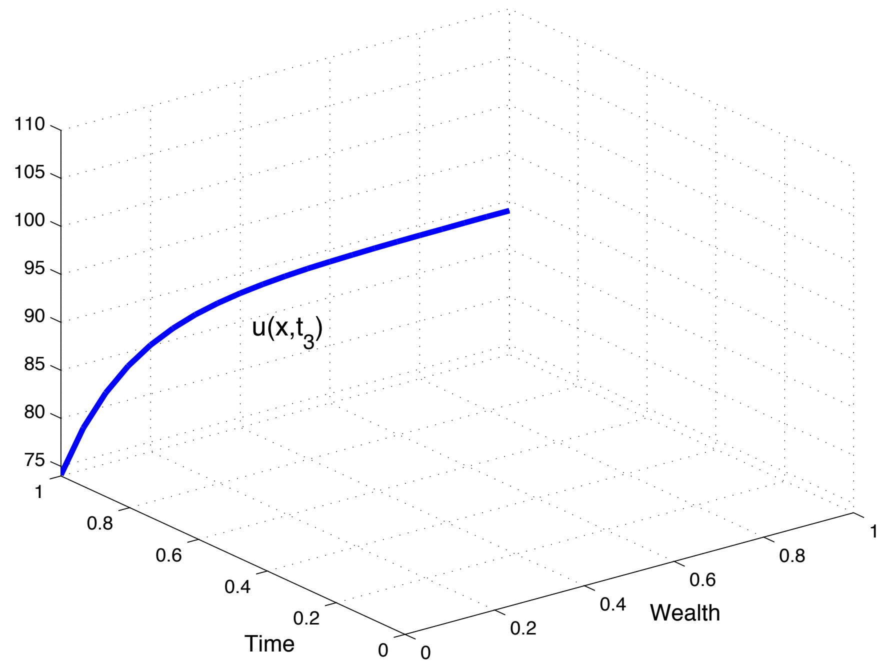
$$\boxed{U(x, t_2) = u(x, A_{t_2}) \in \mathcal{F}_{t_2}}$$

Performance measurement

time t_3, information $\mathcal{F}_{t_3}$

risk premium

$$A_{t_3} = \int_0^{t_3} |\lambda|^2 \, ds \longrightarrow$$



$$A_{t_3} \dashrightarrow + \dashleftarrow u(x, t_3)$$

$$\Downarrow$$

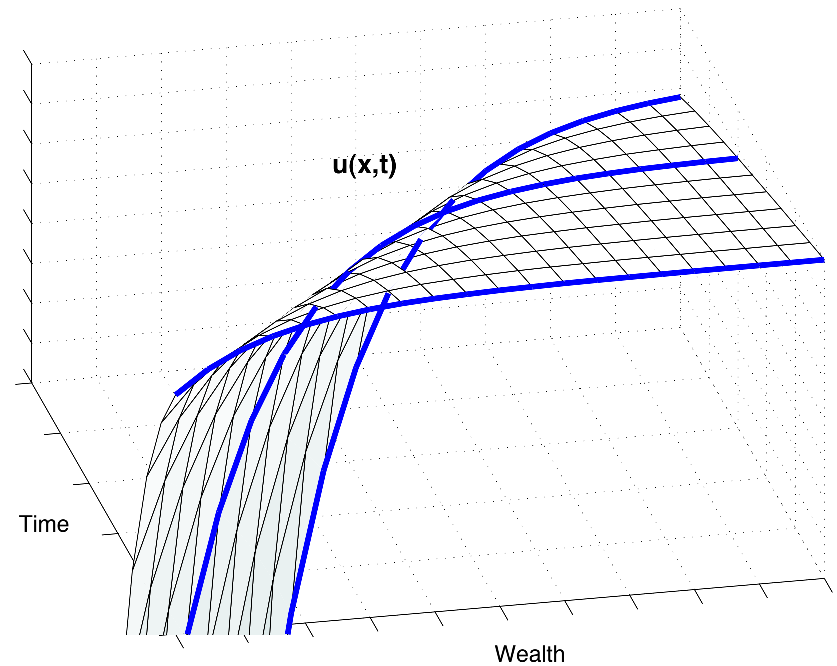
$$\boxed{U(x, t_3) = u(x, A_{t_3}) \in \mathcal{F}_{t_3}}$$

Forward performance measurement

time t, information $\mathcal{F}_t$

market

$\longrightarrow$



$$MI(t) \dashrightarrow + \dashleftarrow u(x,t)$$

$$\Downarrow$$

$$\boxed{U(x,t) = u(x, A_t) \in \mathcal{F}_t}$$

Properties of the performance process

$$U(x,t) = u(x, A_t)$$

- the deterministic risk preferences $u(x,t)$ are compiled with the stochastic market input $A_t = \int_0^t |\lambda|^2 \, ds$

- the evolution of preferences is "deterministic"

- the dynamic risk preferences $u(x,t)$ reflect the risk tolerance and the impatience of the investor

Optimal allocations

Optimal allocations

- Let X_t^* be the optimal wealth, and A_t the time-rescaling processes

$$dX_t^* = \sigma_t \pi_t^* \cdot (\lambda_t dt + dW_t)$$
$$dA_t = |\lambda_t|^2 dt$$

- Define

$$R_t^* \triangleq r(X_t^*, A_t) \qquad\qquad r(x,t) = -\frac{u_x(x,t)}{u_{xx}(x,t)}$$

Optimal portfolios

$$\pi_t^* = \sigma_t^+ \lambda_t R_t^*$$

A system of SDEs at the optimum

$$\begin{cases} dX_t^* = r(X_t^*, A_t)\lambda_t \cdot (\lambda_t \, dt + dW_t) \\ dR_t^* = r_x(X_t^*, A_t) dX_t^* \end{cases}$$

$$\pi_t^* = \sigma_t^+ \lambda_t R_t^*$$

The optimal wealth and portfolios are explicitly constructed

if the function $r(x,t)$ is known

Concave utility inputs and increasing harmonic functions

Concave utility inputs and increasing harmonic functions

There is a one-to-one correspondence between strictly concave solutions $u(x, t)$ to

$$u_t = \frac{1}{2} \frac{u_x^2}{u_{xx}}$$

and strictly increasing solutions to

$$h_t + \frac{1}{2} h_{xx} = 0$$

Concave utility inputs and increasing harmonic functions

- Increasing harmonic function $h : \mathbb{R} \times [0, +\infty) \to \mathbb{R}$ is represented as

$$h(x,t) = \int_{\mathbb{R}} \frac{e^{yx - \frac{1}{2}y^2 t} - 1}{y} \nu(dy)$$

- The associated utility input $u : \mathbb{R} \times [0, +\infty) \to \mathbb{R}$ is then given by the concave function

$$u(x,t) = -\frac{1}{2}\int_0^t e^{-h^{(-1)}(x,s)+\frac{s}{2}} h_x\left(h^{(-1)}(x,s), s\right) ds + \int_0^x e^{-h^{(-1)}(z,0)} dz$$

The support of the measure ν plays a key role in the form of the range of h and, as a result, in the form of the domain and range of u as well as in its asymptotic behavior (Inada conditions)

Examples

Measure ν has compact support

$\nu(dy) = \delta_0$, where δ_0 is a Dirac measure at 0

Then,

$$h(x,t) = \int_{\mathbb{R}} \frac{e^{yx - \frac{1}{2}y^2 t} - 1}{y} \delta_0 = x$$

and

$$u(x,t) = -\frac{1}{2}\int_0^t e^{-x+\frac{s}{2}} ds + \int_0^x e^{-z} dz = 1 - e^{-x+\frac{t}{2}}$$

Measure ν has compact support

$$\nu(dy) = \frac{b}{2}(\delta_a + \delta_{-a}), \quad a, b > 0$$

$\delta_{\pm a}$ is a Dirac measure at $\pm a$

Then,

$$h(x,t) = \frac{b}{a} e^{-\frac{1}{2}a^2 t} \sinh(ax)$$

If, $a = 1$, then

$$u(x,t) = \frac{1}{2}\left(\ln\left(x + \sqrt{x^2 + b^2 e^{-t}}\right) - \frac{e^t}{b^2} x \left(x - \sqrt{x^2 + b^2 e^{-t}}\right) - \frac{t}{2}\right)$$

If $a \neq 1$, then

$$u(x,t) = \frac{(\sqrt{a})^{1+\frac{1}{\sqrt{a}}}}{a-1}\, e^{\frac{1-\sqrt{a}}{2}t}\, \frac{\frac{\beta}{\sqrt{a}} e^{-at} + (1+\sqrt{a}\,)x\left(\sqrt{a}\,x + \sqrt{ax^2 + \beta e^{-at}}\right)}{\left(\sqrt{a}\,x + \sqrt{ax^2 + \beta e^{-at}}\right)^{1+\frac{1}{\sqrt{a}}}}$$

Measure ν has infinite support

$$\nu(dy) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}y^2}\, dy$$

Then

$$h(x,t) = F\left(\frac{x}{\sqrt{t+1}}\right) \qquad\qquad F(x) = \int_0^x e^{\frac{z^2}{2}}\, dz$$

and

$$u(x,t) = F\Big(F^{(-1)}(x) - \sqrt{t+1}\,\Big)$$

Optimal processes and increasing harmonic functions

Optimal processes and risk tolerance

$$\begin{cases} dX_t^* = r(X_t^*, A_t)\lambda_t \cdot (\lambda_t \, dt + dW_t) \\ dR_t^* = r_x(X_t, A_t) \, dX_t^* \end{cases}$$

Local risk tolerance function and fast diffusion equation

$$r_t + \frac{1}{2} r^2 r_{xx} = 0$$

$$r(x,t) = -\frac{u_x(x,t)}{u_{xx}(x,t)}$$

Local risk tolerance and increasing harmonic functions

If $h : \mathbb{R} \times [0, +\infty) \longrightarrow \mathbb{R}$ is an increasing harmonic function then

$r : \mathbb{R} \times [0, +\infty) \longrightarrow \mathbb{R}^+$ given by

$$r\left(x,t\right)=h_x\left(h^{(-1)}\left(x,t\right),t\right)=\int_{\mathbb{R}}e^{yh^{(-1)}(x,t)-\frac{1}{2}y^2t}\nu\left(dy\right)$$

is a risk tolerance function solving the FDE

Optimal portfolio and optimal wealth

- Let h be an increasing solution of the backward heat equation

$$h_t + \frac{1}{2} h_{xx} = 0$$

and $h^{(-1)}$ stands for its spatial inverse

- Let the market input processes A and M by defined by

$$A_t = \int_0^t |\lambda|^2 \, ds \quad \text{and} \quad M_t = \int_0^t \lambda \cdot dW$$

- Then the optimal wealth and optimal portfolio processes are given by

$$X_t^{*,x} = h\left(h^{(-1)}(x,0) + A_t + M_t, A_t\right)$$

and

$$\pi_t^* = h_x\left(h^{(-1)}\left(X_t^{*,x}, A_t\right), A_t\right) \sigma_t^+ \lambda_t$$

Complete construction

Utility inputs and harmonic functions

$$u_t = \frac{1}{2}\,\frac{u_x^2}{u_{xx}} \qquad \Longleftrightarrow \qquad h_t + \frac{1}{2}\,h_{xx} = 0$$

Harmonic functions and positive Borel measures

$$h(x,t) \qquad \Longleftrightarrow \qquad \nu(dy)$$

Optimal wealth process

$$X^{*,x} = h\left(h^{(-1)}(x,0) + A + M, A\right) \qquad M = \int_0^t \lambda \cdot dW_s, \quad \langle M \rangle = A$$

Optimal portfolio process

$$\pi^{*,x} = h_x\left(h^{(-1)}\left(X^{*,x}, A\right), A\right)\sigma^{+}\lambda$$

The measure ν emerges as the defining element

$$\nu \quad \Rightarrow \quad h \quad \Rightarrow \quad u$$

How do we choose ν and what does it represent for the investor's risk attitude?

Inferring investor's preferences

Calibration of risk preferences to the market

Given the desired distributional properties of his/her optimal wealth in a specific market environment, what can we say about the investor's risk preferences?

Investor's investment targets

- Desired future expected wealth
- Desired distribution

References

Sharpe (2006)

Sharpe-Golstein (2005)

Distributional properties of the optimal wealth process

The case of deterministic market price of risk

Using the explicit representation of $X^{*,x}$ we can compute the distribution, density, quantile and moments of the optimal wealth process.

- $$\mathbb{P}\left(X_t^{*,x} \leq y\right) = N\left(\frac{h^{(-1)}(y, A_t) - h^{(-1)}(x,0) - A_t}{\sqrt{A_t}}\right)$$

- $$f_{X_t^{*,x}}(y) = n\left(\frac{h^{(-1)}(y, A_t) - h^{(-1)}(x,0) - A_t}{\sqrt{A_t}}\right)\frac{1}{r(y, A_t)}$$

- $$y_p = h\left(h^{(-1)}(x,0) + A_t + \sqrt{A_t}N^{(-1)}(p), A_t\right)$$

- $$EX_t^{*,x} = h\left(h^{(-1)}(x,0) + A_t, 0\right)$$

Target: The mapping $x \to E\left(X_t^{*,x}\right)$ is linear, for all $x > 0$.

Then, there exists a positive constant $\gamma > 0$ such that the investor's forward performance process is given by

$$U\left(x,t\right) = \frac{\gamma}{\gamma - 1} x^{\frac{\gamma-1}{\gamma}} e^{-\frac{1}{2}(\gamma-1)A_t}, \quad \text{if} \quad \gamma \neq 1$$

and by

$$U_t\left(x\right) = \ln x - \frac{1}{2} A_t, \quad \text{if} \quad \gamma = 1$$

Moreover,

$$E\left(X_t^{*,x}\right) = x e^{\gamma A_t}$$

Calibrating the investor's preferences consists of choosing a time horizon, T, and the level of the mean, mx $(m > 1)$.Then, the corresponding γ must solve $x e^{\gamma A_T} = mx$ and, thus, is given by

$$\gamma = \frac{\ln m}{A_T}$$

The investor can calibrate his expected wealth only for a single time horizon.

Relaxing the linearity assumption

- The linearity of the mapping $x \to E\left(X_t^{*,x}\right)$ is a very strong assumption. It only allows for calibration of a single parameter, namely, the slope, and only at a single time horizon.

- Therefore, if one intends to calibrate the investor's preferences to more refined information, then one needs to accept a more complicated dependence of $E\left(X_t^{*,x}\right)$ on x.

Target: Fix x_0 and consider calibration to $E\left(X_t^{*,x_0}\right)$, for $t \geq 0$

The investor then chooses an increasing function $m(t)$ (with $m(t) > 1$) to represent $E\left(X_t^{*,x_0}\right)$,

$$E\left(X_t^{*,x_0}\right) = m(t), \quad \text{for } t \geq 0.$$

- What does it say about his preferences?

- Moreover, can he choose an arbitrary increasing function $m(t)$?

Relaxing the linearity assumption

For simplicity, assume $x_0 = 1$ and that ν is a probability measure. Then, $h^{(-1)}(1,0) = 0$ and we deduce that

$$E\left(X_t^{*,1}\right) = h\left(A_t, 0\right) = \int_0^\infty e^{yA_t} \nu\left(dy\right)$$

Clearly, the investor may only specify the function $m\left(t\right)$, $t > 0$, which can be represented, for **some** probability measure ν in the form

$$m\left(t\right) = \int_0^\infty e^{yA_t} \nu\left(dy\right)$$

Conclusions

- Space-time monotone investment performance criteria
- Explicit construction of forward performance process
- Connection with space-time harmonic functions
- Explicit construction of the optimal wealth and optimal portfolio processes
- The "trace" measure as the defining element of the entire construction
- Calibration of the trace to the market
- Inference of dynamic risk preferences