A new approach for investment performance measurement

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Performance measurement of investment strategies
Market environment

Riskless and risky securities

- $(\Omega, \mathcal{F}, \mathbb{P})$ ; $W = (W^1, \ldots, W^d)$ standard Brownian Motion

- Traded securities

\[
\begin{aligned}
1 \leq i \leq k \\
\begin{cases}
    dS^i_t &= S^i_t \left( \mu^i_t dt + \sigma^i_t \cdot dW^i_t \right), \\
    dB_t &= r_t B_t dt, \\
\end{cases}
\end{aligned}
\]

$s_0 > 0$, $B_0 = 1$

- No assumptions on market completeness

$\mu_t, r_t \in \mathbb{R}$, $\sigma^i_t \in \mathbb{R}^d$ bounded and $\mathcal{F}_t$-measurable stochastic processes

- Postulate existence of an $\mathcal{F}_t$-measurable stochastic process $\lambda_t \in \mathbb{R}^d$

satisfying

\[
\mu_t - r_t \mathbf{1} = \sigma^T_t \lambda_t
\]

- No assumptions on market completeness
Market environment

- Self-financing investment strategies \( \pi_0^t, \pi_t = (\pi_1^t, \ldots, \pi_i^t, \ldots, \pi_k^t) \)

- Present value of this allocation

\[
X_t = \sum_{i=0}^{k} \pi_i^t
\]

\[
dX_t = \sum_{i=1}^{k} \pi_i^t \sigma_i^t \cdot (\lambda_t \ dt + dW_t)
\]

\[
= \sigma_t \pi_t \cdot (\lambda_t \ dt + dW_t)
\]
Traditional framework

A (deterministic) utility datum $u_T(x)$ is assigned at the end of a fixed investment horizon

$$U_T(x) = u_T(x)$$

No market input to the choice of terminal utility

Backwards in time generation of the indirect utility

$$V(x, s; T) = \sup_\pi E_P(u_T(X_T^\pi) | \mathcal{F}_s; X_s^\pi = x)$$

$$V(x, s; T) = \sup_\pi E_P(V(X_t^\pi, t; T) | \mathcal{F}_s; X_s^\pi = x) \quad \text{(DPP)}$$

$$V(x, s; T) = E_P(V(X_t^{\pi^*}, t; T) | \mathcal{F}_s; X_s^{\pi^*} = x)$$

The value function process becomes the intermediate utility for all $t \in [0, T)$
The value function process

\[ V(x, s; T) \in \mathcal{F}_s \quad V(x, t; T) \in \mathcal{F}_t \quad u_T(x) \in \mathcal{F}_0 \]

0 \quad s \quad t \quad T

- For each self-financing strategy, represented by \( \pi \), the associated wealth \( X_t^\pi \) satisfies

\[
E_P(V(X_t^\pi, t; T)|\mathcal{F}_s) \leq V(X_s^\pi, s; T), \quad 0 \leq s \leq t \leq T
\]

- There exists a self-financing strategy, represented by \( \pi^* \), for which the associated wealth \( X_t^{\pi^*} \) satisfies

\[
E_P(V(X_t^{\pi^*}, t; T)|\mathcal{F}_s) = V(X_s^{\pi^*}, s; T), \quad 0 \leq s \leq t \leq T
\]

- At expiration, \( V(x, T; T) = u_T(x) \in \mathcal{F}_0 \)
Study of the value function process

- “Arbitrary” environments
  - Duality methods
  - Martingale representation results

- Markovian environments
  - HJB equation
  - Feedback optimal controls
  - Weak solutions
A stochastic PDE for the value function process
Intuition

- Assume that, for \( t \in [0, T] \), the value function \( V(x, t) \) solves
  \[
  dV(x, t) = b(x, t) \, dt + a(x, t) \cdot dW_t
  \]
  where \( b, a \) are \( F_t \)-measurable processes.

- Recall that for an arbitrary admissible portfolio \( \pi \), the associated wealth process, \( X_\pi^\pi \), solves
  \[
  dX_\pi^\pi = \sigma_\pi \pi_t (\lambda_t \, dt + dW_t)
  \]

- Applying the Itô-Ventzell formula to \( V(X_\pi^\pi, t) \) yields
  \[
  dV(X_\pi^\pi, t) = b(X_\pi^\pi, t) \, dt + a(X_\pi^\pi, t) \cdot dW_t
  \]
  
  \[
  + V_x(X_\pi^\pi, t) \, dX_\pi^\pi + \frac{1}{2} V_{xx}(X_\pi^\pi, t) \, d\langle X_\pi^\pi \rangle_t + a_x(X_\pi^\pi, t) \cdot d\langle W, X_\pi^\pi \rangle_t
  \]
  
  \[
  = \left( b(X_\pi^\pi, t) + V_x(X_\pi^\pi, t) \, \sigma_\pi \pi_t \, \lambda_t + \sigma_\pi \pi_t \cdot a_x(X_\pi^\pi, t) + \frac{1}{2} V_{xx}(X_\pi^\pi, t) \, |\sigma_\pi \pi_t|^2 \right) \, dt
  \]
  
  \[
  + (a(X_\pi^\pi, t) + V_x(X_\pi^\pi, t) \, \sigma_\pi \pi_t) \cdot dW_t
  \]
Intuition (continued)

• By the monotonicity and concavity assumptions, the quantity

\[
\sup_{\pi} \left( V_x (X_t^\pi, t) \sigma_t \pi_t \cdot \lambda_t + \sigma_t \pi_t \cdot a_x (X_t^\pi, t) + \frac{1}{2} V_{xx} (X_t^\pi, t) \sigma_t \pi_t^2 \right)
\]

is well defined

• Calculating the optimum \( \pi^* \) yields

\[
\pi_t^* = -\sigma_t^+ \frac{V_x (X_t^{\pi^*}, t) \lambda_t + a_x (X_t^{\pi^*}, t)}{V_{xx} (X_t^{\pi^*}, t)}
\]

• Deduce that the above supremum is given by

\[
M^* (X_t^{\pi^*}, t) = -\frac{\sigma_t \sigma_t^+ (V_x (X_t^{\pi^*}, t) \lambda_t + a_x (X_t^{\pi^*}, t))}{2V_{xx} (X_t^{\pi^*}, t)}^2
\]

• The drift coefficient \( b \) must satisfy

\[
b (X_t^{\pi^*}, t) = -M^* (X_t^{\pi^*}, t)
\]
SPDE for the value function process

- Market \((\sigma_t, \sigma_t^+, \lambda_t)\); volatility \(a(x, t) \in \mathcal{F}_t\)

\[
dV = \frac{1}{2} \frac{|\sigma \sigma^+ \mathcal{A}(V \lambda + a)|^2}{\mathcal{A}^2 V} \, dt + a \cdot dW
\]

\[
V(x, T) = u_T(x) \in \mathcal{F}_0 ; \quad \mathcal{A} = \frac{\partial}{\partial x}
\]

- Feedback optimal portfolio vector

\[
\pi_t^* = \pi^*(X_t^{\pi, *}, t) = -\sigma + \frac{\mathcal{A}(V \lambda + a)}{\mathcal{A}^2 V}(X_t^{\pi, *}, t)
\]

- Choices for the volatility process \(a\)?
A Markovian example

- \( r_t = r(Y_t), \quad \mu_t = \mu(Y_t), \quad \sigma_t = \sigma(Y_t) \)

\[ dY_t = \theta(Y_t) \, dt + \Theta^T(Y_t) \, dW_t \]

- Value function

\[ v(x, y, t, T) = \sup \pi \mathbb{E} \left( u_T(X_T^\pi) \mid X_t^\pi = x, Y_t = y \right) \]

- HJB equation

\[ v_t + \sup \pi \left( \frac{1}{2} |\sigma \pi|^2 v_{xx} + \sigma \pi \cdot \sigma \sigma^+ (\lambda v_x + \Theta v_{xy}) + \frac{1}{2} \Theta^T \Theta \cdot v_{yy} + \theta \cdot v_y \right) \]

\[ = v_t - \frac{1}{2} \frac{|\sigma \sigma^+(v_x \lambda + \Theta v_{xy})|^2}{v_{xx}} + \frac{1}{2} \Theta^T \Theta \cdot v_{yy} + \theta \cdot v_y = 0 \]
• The SPDE for the value function process

\[ V(x, t) = v(x, Y_t, t; T) \]

\[ dV(x, t) = v_t \, dt + v_y \cdot dY + \frac{1}{2} v_{yy} \cdot d\langle Y \rangle \]

\[ \text{HJB} \equiv \left( \frac{1}{2} \frac{\sigma \sigma^+(v_x \lambda + \Theta v_{xy})^2}{v_{xx}} - \frac{1}{2} \Theta^T \Theta \cdot v_{yy} - \theta \cdot v_y \right) \, dt \]

\[ + v_y \cdot \left( \theta \, dt + \Theta^T \, dW \right) + \frac{1}{2} v_{yy} \cdot \Theta^T \Theta \, dt \]

\[ = \frac{1}{2} \frac{\sigma \sigma^+(V_x \lambda + a_x(x, t))^2}{V_{xx}} \, dt + a(x, t) \cdot dW \]

• The volatility process is uniquely determined: \( a(x, t) = \Theta v_y(x, Y_t, t; T) \)
Going beyond the deterministic terminal utility problem
Motivation (partial)

- Terminal utility might be $\omega$-dependent

Liability management, indifference valuation

$$u_T(x, \omega) = -e^{-\gamma(x-C_T(\omega))}; \quad C_T \in \mathcal{F}_T$$

Numeraire consistency

$$u_T(x, \omega) = -e^{-\gamma_T(\omega)x}$$

- Need to extend the value function process beyond $T$

- Need to manage liabilities of arbitrary maturities

How do we formulate investment performance criteria?
**Investment performance process**

\[ U(x, t) \] is an \( \mathcal{F}_t \)-adapted process, \( t \geq 0 \)

- The mapping \( x \rightarrow U(x, t) \) is increasing and concave

- For each self-financing strategy, represented by \( \pi \), the associated (discounted) wealth \( X_t^{\pi} \) satisfies

\[
E_{\mathbb{P}}(U(X_t^{\pi}, t) \mid \mathcal{F}_s) \leq U(X_s^{\pi}, s), \quad 0 \leq s \leq t
\]

- There exists a self-financing strategy, represented by \( \pi^* \), for which the associated (discounted) wealth \( X_t^{\pi^*} \) satisfies

\[
E_{\mathbb{P}}(U(X_t^{\pi^*}, t) \mid \mathcal{F}_s) = U(X_s^{\pi^*}, s), \quad 0 \leq s \leq t
\]
Optimality across times

\[ U(x, t) \in \mathcal{F}_t \]

\[ U(x, s) \in \mathcal{F}_s \quad U(x, t) \in \mathcal{F}_t \]

\[ U(x, s) = \sup_{\mathcal{A}} E(U(X_t^\pi, t)|\mathcal{F}_s, X_s = x) \]

- What is the meaning of this process?
- Does such a process always exist?
- Is it unique?
Forward performance process

A datum \( u_0(x) \) is assigned at the beginning of the trading horizon, \( t = 0 \)

\[
U(x, 0) = u_0(x)
\]

Forward in time criteria

\[
\begin{align*}
E_{\mathbb{P}}(U(X^\pi_t, t)|\mathcal{F}_s) & \leq U(X^\pi_s, s), & 0 \leq s \leq t \\
E_{\mathbb{P}}(U(X^\pi^*_t, t)|\mathcal{F}_s) & = U(X^\pi^*_s, s), & 0 \leq s \leq t
\end{align*}
\]

Many difficulties due to “inverse in time” nature of the problem
The forward performance SPDE
The forward performance SPDE

Let $U(x, t)$ be an $\mathcal{F}_t$–measurable process such that the mapping $x \rightarrow U(x, t)$ is increasing and concave. Let also $U = U(x, t)$ be the solution of the stochastic partial differential equation

$$dU = \frac{1}{2} \frac{|\sigma \sigma^+ A (U \lambda + a)|^2}{A^2 U} dt + a \cdot dW$$

where $a = a(x, t)$ is an $\mathcal{F}_t$–adapted process, while $A = \frac{\partial}{\partial x}$.

Then $U(x, t)$ is a forward performance process.

The process $a$ may depend on $t, x, U$, its spatial derivatives etc.
Optimal portfolios and wealth

At the optimum

- The optimal portfolio vector \( \pi^* \) is given in the feedback form

\[
\pi^*_t = \pi^*(X^*_t, t) = -\sigma + \frac{A(U \lambda + a)}{A^2 U}(X^*_t, t)
\]

- The optimal wealth process \( X^* \) solves

\[
dX^*_t = -\sigma \sigma + \frac{A(U \lambda + a)}{A^2 U}(X^*_t, t) (\lambda dt + dW_t)
\]
Solutions to the forward performance SPDE

\[ dU = \frac{1}{2} \left| \sigma \sigma^+ A (U \lambda + a) \right|^2 dt + a \cdot dW \]

Local differential coefficients

\[ a (x, t) = F (x, t, U (x, t), U_x (x, t)) \]

Difficulties

- The equation is fully nonlinear
- The diffusion coefficient depends, in general, on \( U_x \) and \( U_{xx} \)
- The equation is not (degenerate) elliptic
The zero volatility case: \( a(x, t) \equiv 0 \)
**Space-time monotone performance process**

The forward performance SPDE simplifies to

\[ dU = \frac{1}{2} \left| \sigma \sigma^+ \mathcal{A} (U \lambda) \right|^2 dt \]

The process

\[ U (x, t) = u (x, A_t) \quad \text{with} \quad A_t = \int_0^t \left| \sigma_s \sigma^+_s \lambda_s \right|^2 ds \]

with \( u : \mathbb{R} \times [0, +\infty) \rightarrow \mathbb{R} \), increasing and concave with respect to \( x \), and solving

\[ u_t u_{xx} = \frac{1}{2} u_x^2 \]

is a solution.

MZ (2006)

Berrier, Rogers and Tehranchi (2007)
Performance measurement

time $t_1$, information $\mathcal{F}_{t_1}$

Risk premium

$$A_{t_1} = \int_0^{t_1} |\lambda|^2 \, ds$$

Wealth Time

$$U(x, t_1) = u(x, A_{t_1}) \in \mathcal{F}_{t_1}$$
Performance measurement

time $t_2$, information $\mathcal{F}_{t_2}$

risk premium

$$A_{t_2} = \int_0^{t_2} |\lambda|^2 \, ds$$

$$U(x, t_2) = u(x, A_{t_2}) \in \mathcal{F}_{t_2}$$
Performance measurement

time $t_3$, information $\mathcal{F}_{t_3}$

risk premium

$A_{t_3} = \int_0^{t_3} |\lambda|^2 ds$

$A_{t_3} \rightarrow + \leftarrow u(x, t_3)$

$U(x, t_3) = u(x, A_{t_3}) \in \mathcal{F}_{t_3}$
Forward performance measurement

(time $t$, information $\mathcal{F}_t$)

$\mathcal{F}_t$

market

\[ MI(t) \quad \longrightarrow \quad + \quad \longleftarrow \quad u(x, t) \]

\[ U(x, t) = u(x, A_t) \in \mathcal{F}_t \]
Properties of the performance process

\[ U(x, t) = u(x, A_t) \]

- the deterministic risk preferences \( u(x, t) \) are compiled with the stochastic market input \( A_t = \int_0^t |\lambda|^2 \, ds \)
- the evolution of preferences is “deterministic”
- the dynamic risk preferences \( u(x, t) \) reflect the risk tolerance and the impatience of the investor
Optimal allocations
Optimal allocations

• Let \( X_t^* \) be the optimal wealth, and \( A_t \) the time-rescaling processes

\[
dX_t^* = \sigma_t \pi_t^* \cdot (\lambda_t dt + dW_t)
\]

\[
dA_t = |\lambda_t|^2 dt
\]

• Define

\[
R_t^* \triangleq r(X_t^*, A_t) \quad r(x, t) = -\frac{u_x(x, t)}{u_{xx}(x, t)}
\]

Optimal portfolios

\[
\pi_t^* = \sigma_t^+ \lambda_t R_t^*
\]
A system of SDEs at the optimum

\[
\begin{align*}
    dX_t^* &= r(X_t^*, A_t)\lambda_t \cdot (\lambda_t \ dt + dW_t) \\
    dR_t^* &= r_x(X_t^*, A_t)dX_t^* \\
    \pi_t^* &= \sigma_t^+ \lambda_t R_t^*
\end{align*}
\]

The optimal wealth and portfolios are explicitly constructed

if the function \( r(x, t) \) is known
Concave utility inputs and increasing harmonic functions
Concave utility inputs and increasing harmonic functions

There is a one-to-one correspondence between strictly concave solutions $u(x, t)$ to

$$u_t = \frac{1}{2} \frac{u_x^2}{u_{xx}}$$

and strictly increasing solutions to

$$h_t + \frac{1}{2} h_{xx} = 0$$
Concave utility inputs and increasing harmonic functions

- Increasing harmonic function $h : \mathbb{R} \times [0, +\infty) \rightarrow \mathbb{R}$ is represented as

$$h(x, t) = \int_{\mathbb{R}} \frac{e^{yx} - \frac{1}{2}y^2 t - 1}{y} \nu(dy)$$

- The associated utility input $u : \mathbb{R} \times [0, +\infty) \rightarrow \mathbb{R}$ is then given by the concave function

$$u(x, t) = -\frac{1}{2} \int_{0}^{t} e^{-h(-1)(x,s)} + \frac{s}{2} h_x \left( h^{-1}(x, s), s \right) ds + \int_{0}^{x} e^{-h(-1)(z,0)} dz$$

The support of the measure $\nu$ plays a key role in the form of the range of $h$ and, as a result, in the form of the domain and range of $u$ as well as in its asymptotic behavior (Inada conditions)
Examples
Measure $\nu$ has compact support

$$\nu(dy) = \delta_0,$$ where $\delta_0$ is a Dirac measure at 0

Then,

$$h(x, t) = \int_{\mathbb{R}} \frac{eyx - \frac{1}{2}y^2t - 1}{y} \delta_0 = x$$

and

$$u(x, t) = -\frac{1}{2} \int_0^t e^{-x+\frac{s}{2}} ds + \int_0^x e^{-z} dz = 1 - e^{-x+\frac{t}{2}}$$
Measure $\nu$ has compact support

$\nu(dy) = \frac{b}{2}(\delta_a + \delta_{-a}), \quad a, b > 0$

$\delta_{\pm a}$ is a Dirac measure at $\pm a$

Then,

$h(x, t) = \frac{b}{a}e^{-\frac{1}{2}a^2t} \sinh(ax)$

If, $a = 1$, then

$u(x, t) = \frac{1}{2}\left(\ln\left(x + \sqrt{x^2 + b^2e^{-t}}\right) - \frac{e^t}{b^2}x\left(x - \sqrt{x^2 + b^2e^{-t}}\right) - \frac{t}{2}\right)$

If $a \neq 1$, then

$u(x, t) = \frac{\left(\sqrt{a}\right)^{1+\frac{1}{a}}}{a-1} e^{\frac{1-\sqrt{a}}{2}t} \frac{\beta e^{-at} + (1 + \sqrt{a})x \left(\sqrt{a}x + \sqrt{ax^2 + \beta e^{-at}}\right)}{\left(\sqrt{a}x + \sqrt{ax^2 + \beta e^{-at}}\right)^{1+\frac{1}{a}}}$
Measure $\nu$ has infinite support

$$
\nu(dy) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} dy
$$

Then

$$
h(x, t) = F \left( \frac{x}{\sqrt{t} + 1} \right)
\quad F(x) = \int_0^x e^{\frac{z^2}{2}} \, dz
$$

and

$$
u(x, t) = F \left( F^{(-1)}(x) - \sqrt{t + 1} \right)
$$
Optimal processes and increasing harmonic functions
Optimal processes and risk tolerance

\begin{align*}
\begin{cases}
\d X_t^* &= r(X_t^*, A_t)\lambda_t \cdot (\lambda_t \, dt + dW_t) \\
\d R_t^* &= r_x(X_t, A_t) \, dX_t^* 
\end{cases}
\end{align*}

Local risk tolerance function and fast diffusion equation

\begin{align*}
\frac{1}{2} r^2 r_{xx} &= 0 \\
 r(x, t) &= -\frac{u_x(x, t)}{u_{xx}(x, t)}
\end{align*}
Local risk tolerance and increasing harmonic functions

If \( h : \mathbb{R} \times [0, +\infty) \rightarrow \mathbb{R} \) is an increasing harmonic function then

\[
    r : \mathbb{R} \times [0, +\infty) \rightarrow \mathbb{R}^+ \quad \text{given by}
\]

\[
    r(x, t) = h_x \left( h^{(-1)}(x, t), t \right) = \int_{\mathbb{R}} e^{y h^{(-1)}(x, t)} \frac{1}{2} y^2 t \nu \, (dy)
\]

is a risk tolerance function solving the FDE.
Optimal portfolio and optimal wealth

- Let \( h \) be an increasing solution of the backward heat equation

\[
ht + \frac{1}{2} h_{xx} = 0
\]

and \( h^{(-1)} \) stands for its spatial inverse

- Let the market input processes \( A \) and \( M \) be defined by

\[
A_t = \int_0^t |\lambda|^2 ds \quad \text{and} \quad M_t = \int_0^t \lambda \cdot dW
\]

- Then the optimal wealth and optimal portfolio processes are given by

\[
X^{*,x}_t = h \left( h^{(-1)}(x,0) + A_t + M_t, A_t \right)
\]

and

\[
\pi^*_t = h_x \left( h^{(-1)} \left( X^{*,x}_t, A_t \right), A_t \right) \sigma_t^+ \lambda_t
\]
Complete construction

Utility inputs and harmonic functions

\[ u_t = \frac{1}{2} \frac{u_x^2}{u_{xx}} \quad \iff \quad h_t + \frac{1}{2} h_{xx} = 0 \]

Harmonic functions and positive Borel measures

\[ h(x, t) \quad \iff \quad \nu(dy) \]

Optimal wealth process

\[ X^{*,x} = h(h^{-1}(x, 0) + A + M, A) \quad M = \int_0^t \lambda \cdot dW_s, \quad \langle M \rangle = A \]

Optimal portfolio process

\[ \pi^{*,x} = h_x(h^{-1}(X^{*,x}, A), A) \sigma^+ \lambda \]

The measure \( \nu \) emerges as the defining element

\[ \nu \quad \Rightarrow \quad h \quad \Rightarrow \quad u \]

How do we choose \( \nu \) and what does it represent for the investor’s risk attitude?
Inferring investor’s preferences
Calibration of risk preferences to the market

Given the desired distributional properties of his/her optimal wealth in a specific market environment, what can we say about the investor’s risk preferences?

Investor’s investment targets

- Desired future expected wealth
- Desired distribution

References

Sharpe (2006)
Sharpe-Golstein (2005)
Distributional properties of the optimal wealth process

The case of deterministic market price of risk

Using the explicit representation of $X^{*},x$ we can compute the distribution, density, quantile and moments of the optimal wealth process.

- \[ P \left( X^{*},x_t \leq y \right) = N \left( \frac{h^{-1}(y, A_t) - h^{-1}(x, 0) - A_t}{\sqrt{A_t}} \right) \]

- \[ f_{X^{*},x_t}(y) = n \left( \frac{h^{-1}(y, A_t) - h^{-1}(x, 0) - A_t}{\sqrt{A_t}} \right) \frac{1}{r(y, A_t)} \]

- \[ y_p = h \left( h^{-1}(x, 0) + A_t + \sqrt{A_t} N^{-1}(p), A_t \right) \]

- \[ EX^{*},x_t = h \left( h^{-1}(x, 0) + A_t, 0 \right) \]
The mapping $x \mapsto E\left(X_t^*,x\right)$ is linear, for all $x > 0$.

Then, there exists a positive constant $\gamma > 0$ such that the investor's forward performance process is given by

$$U(x, t) = \frac{\gamma}{\gamma - 1} \frac{x^{\gamma - 1}}{\gamma} e^{-\frac{1}{2}(\gamma - 1)A_t}, \text{ if } \gamma \neq 1$$

and by

$$U_t(x) = \ln x - \frac{1}{2}A_t, \text{ if } \gamma = 1$$

Moreover,

$$E\left(X_t^*,x\right) = xe^{\gamma A_t}$$

Calibrating the investor's preferences consists of choosing a time horizon, $T$, and the level of the mean, $mx$ ($m > 1$). Then, the corresponding $\gamma$ must solve $xe^{\gamma AT} = mx$ and, thus, is given by

$$\gamma = \frac{\ln m}{A_T}$$

The investor can calibrate his expected wealth only for a single time horizon.
Relaxing the linearity assumption

- The linearity of the mapping \( x \rightarrow E \left( X_t^*, x \right) \) is a very strong assumption. It only allows for calibration of a single parameter, namely, the slope, and only at a single time horizon.

- Therefore, if one intends to calibrate the investor’s preferences to more refined information, then one needs to accept a more complicated dependence of \( E \left( X_t^*, x \right) \) on \( x \).

**Target:** Fix \( x_0 \) and consider calibration to \( E \left( X_t^*, x_0 \right) \), for \( t \geq 0 \)

The investor then chooses an increasing function \( m(t) \) (with \( m(t) > 1 \)) to represent \( E \left( X_t^*, x_0 \right) \),

\[
E \left( X_t^*, x_0 \right) = m(t), \quad \text{for } t \geq 0.
\]

- What does it say about his preferences?

- Moreover, can he choose an arbitrary increasing function \( m(t) \)?
Relaxing the linearity assumption

For simplicity, assume $x_0 = 1$ and that $\nu$ is a probability measure. Then, $h^{(-1)}(1, 0) = 0$ and we deduce that

$$E \left( X_{t^*}, 1 \right) = h(A_t, 0) = \int_0^\infty e^{yA_t} \nu(dy)$$

Clearly, the investor may only specify the function $m(t), t > 0$, which can be represented, for some probability measure $\nu$ in the form

$$m(t) = \int_0^\infty e^{yA_t} \nu(dy)$$
Conclusions

• Space-time monotone investment performance criteria

• Explicit construction of forward performance process

• Connection with space-time harmonic functions

• Explicit construction of the optimal wealth and optimal portfolio processes

• The “trace” measure as the defining element of the entire construction

• Calibration of the trace to the market

• Inference of dynamic risk preferences