EQUITY MARKET STABILITY

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Model for an equity market

Equity market framework with capitalizations modeled by continuous semimartingales; e.g., Bachelier, Samuelson...

\[
\frac{dX_i(t)}{X_i(t)} = \alpha_i(t)\, dt + \sum_{\nu=1}^{\mathcal{N}} \xi_{i\nu}(t) \, dW_{\nu}(t), \quad i = 1, \ldots, \mathcal{N},
\]

where \( W(\cdot) = (W_1(\cdot), \cdots, W_n(\cdot))' \) is B.M. in dimension \( \mathcal{N} \geq N \).

- **Mean Return rates** \( \alpha(\cdot) = (\alpha_1(\cdot), \ldots, \alpha_N(\cdot))' \)
- **Volatility rates** \( \xi(\cdot) = (\xi_{i\nu}(\cdot))_{1 \leq i \leq \mathcal{N}, 1 \leq \nu \leq \mathcal{N}} \)
- **Covariance rates** \( \sigma(\cdot) = \xi(\cdot)\xi'(\cdot), \) \((N \times N)\) matrix given by

\[
\sigma_{ij}(t) := \sum_{\nu=1}^{\mathcal{N}} \xi_{i\nu}(t)\xi_{j\nu}(t) = \frac{1}{X_i(t)X_j(t)} \cdot \frac{d}{dt} \langle X_i, X_j \rangle(t).
\]

Both \( \alpha(\cdot) \) and \( \sigma(\cdot) \) are assumed to be locally integrable.
Logarithmic representation

Define log capitalizations $Y(\cdot)$ by

$$Y_i(\cdot) := \log X_i(\cdot),$$

so that the equation

$$\frac{dX_i(t)}{X_i(t)} = \alpha_i(t) \, dt + \sum_{\nu=1}^{N} \xi_{i\nu}(t) \, dW_\nu(t)$$

becomes

$$dY_i(t) = \gamma_i(t) \, dt + \sum_{\nu=1}^{N} \xi_{i\nu}(t) \, dW_\nu(t),$$

where the growth rate of the $i^{th}$ asset is

$$\gamma_i(t) := \alpha_i(t) - \frac{1}{2} \sigma_{ii}(t).$$
Market weights

Define the total market capitalization

\[ X(t) := X_1(t) + \cdots + X_N(t). \]

Consider now the relative weights of the various assets, in terms of capitalization:

\[ \mu_i(t) := \frac{X_i(t)}{X(t)} = \frac{X_i(t)}{X_1(t) + \cdots + X_N(t)}, \quad i = 1, \ldots, N. \]

These are strictly positive numbers, add up to 1, and are determined on the basis of the capitalizations of the market’s various assets at any given time.

They are perfectly observable, at all times.
Ranked log capitalizations

Descending order statistics:

\[
\max_{1 \leq i \leq N} Y_i(\cdot) =: Y_{(1)}(\cdot) \geq \cdots \geq Y_{(N)}(\cdot) := \min_{1 \leq i \leq N} Y_i(\cdot).
\]

Ties are resolved lexicographically, by resorting to the lowest index (name) \( i \).

- Similar ordering for the capitalizations

\[
X_{(k)}(\cdot) = \exp(Y_{(k)}(\cdot))
\]

and for the ordered market weights

\[
\mu_{(k)}(\cdot) = \frac{X_{(k)}(\cdot)}{X_1(t) + \ldots + X_N(t)}, \quad k = 1, \ldots, N.
\]
Capital distribution, U.S. equity market 1973-2013

Market Weight

Stocks Ranked by Capitalization

- 1973
- 1983
- 1993
- 2003
- 2013
Implications of market stability

Many rules-based strategies ("smart beta") are seen to outperform cap-weighted indexes over the long term. Equity market stability is critical to this outperformance.

Example: a continuously rebalanced equal-weight portfolio $E$. (N.B. over $10B$ invested in equal-weight portfolios.) Its relative value process versus the market $M$ is (Fernholz 2002):

$$d \log(V_E(t)/V_M(t)) = dS(t) + \theta(t) \, dt$$

where $\theta(\cdot)$ is a.s. positive and

$$S(t) := \frac{1}{N} \sum_{i=1}^{N} \log \mu_i(t).$$

A stable capital distribution implies that $S(\cdot)$ is mean-reverting, hence the relative performance is positive over sufficiently long time periods.
Collision local times

The ordered processes \( Y_{(k)} \) are themselves semimartingales (as we shall see), and for \( 1 \leq k < \ell \leq N \), the gap

\[
G^{(k,\ell)}(\cdot) := Y_{(k)}(\cdot) - Y_{(\ell)}(\cdot) = \log \left( \frac{X_{(k)}(\cdot)}{X_{(\ell)}(\cdot)} \right) = \log \left( \frac{\mu_{(k)}(\cdot)}{\mu_{(\ell)}(\cdot)} \right) \geq 0
\]

is a continuous, nonnegative semimartingale. We shall denote by

\[
\Lambda^{(k,\ell)}(\cdot) := L^{G^{(k,\ell)}}(\cdot)
\]

its local time at the origin (collision local time of order \( \ell - k + 1 \)).

- For a nonnegative continuous semimartingale \( G(\cdot) \), recall its local time process

\[
L^{G}(\cdot) = \int_{0}^{\cdot} \mathbf{1}_{\{G(t)=0\}} \, dG(t) = \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_{0}^{\cdot} \mathbf{1}_{\{0 \leq G(t) < \varepsilon\}} \, d\langle G \rangle(t).
\]
Reasonable conditions for stability

A: No single stock “crashes” relative to the entire market (coherence):

\[ \lim_{T \to \infty} \frac{1}{T} \log \mu_i(T) = 0, \text{ a.s., } \forall \ i = 1, \ldots, N. \]

B: Convergence in distribution:

\( (\mu(1)(T), \ldots, \mu(N)(T)) \longrightarrow (M_1, \cdots, M_N), \text{ as } T \to \infty. \)

C: Strong laws of large numbers for local times:

\[ \lim_{T \to \infty} \frac{1}{2T} \Lambda^{(k,k+1)}(T) := \lambda_k > 0, \text{ a.s., } \forall \ k = 1, \ldots, N. \]
A result of Fernholz

Suppose that the eigenvalues of $\sigma(\cdot)$ are bounded from above and below a.s., and that $\gamma_i(t) = \gamma(t)$ for all $i = 1, \ldots, n$ (common growth rate). Then

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \mu(1)(t) \, dt = 1.$$  

This is a degenerate form of stability in which a single stock occupies virtually the entire market capitalization.

- This is not consistent with empirical observations.
- We seek a simple model with stability but not concentration into a subset of the stocks.
First-order models

With given real numbers $\gamma, \gamma_1, \ldots, \gamma_N$ and $\sigma_1 > 0, \ldots, \sigma_N > 0$, and independent Brownian motions $W_1(\cdot), \ldots, W_N(\cdot)$, we consider the system of SDEs with rank-based interactions

\[
dY_i(t) = \left( \gamma + \sum_{k=1}^{N} \gamma_k 1\{Y_i(t) = Y_{(k)}(t)\} \right) dt + \sum_{k=1}^{N} \sigma_k 1\{Y_i(t) = Y_{(k)}(t)\} dW_i(t).
\]

Bass & Pardoux (1987) show that this system has a weak solution, unique in distribution (well-posed martingale problem).
Ichiba, Karatzas & Shkolnikov (2013) show that there is pathwise uniqueness, thus also strength, for the above system up until the first time 3 particles collide simultaneously.

- For this never to happen, it is necessary and “almost sufficient” (IKS op. cit.) that the mapping

\[ \{1, \cdots, N\} \ni k \mapsto \sigma_k^2 \in (0, \infty) \] be concave.

- Sarantsev has recently proved that concavity is indeed sufficient.

- However, the weaker property \( \Lambda^{(k, \ell)}(\cdot) \equiv 0 \) for \( \ell - k \geq 2 \) is always satisfied.
In such a system, the ranked log-capitalizations are rather simple to describe:

\[ Y_{(k)}(t) = Y_{(k)}(0) + (\gamma + \gamma_k)t + \sigma_k B_k(t) \]

\[ + \frac{1}{2} \left( \Lambda^{(k,k+1)}(t) - \Lambda^{(k-1,k)}(t) \right). \]

In other words, they behave as independent Brownian motions

\[ B_k(\cdot) := \sum_{i=1}^{N} \int_{0}^{\cdot} 1\{Y_i(t) = Y_{(k)}(t)\} dW_i(t) \]

(by the P. Lévy theorem) with drift, bouncing off each other with the help of the two-by-two collision local times.

Alternative Picture: Brownian motion with drift, reflecting off the walls of the Weyl chamber

\[ \mathcal{W} = \{(y_1, \ldots, y_N) \in \mathbb{R}^N : y_1 \geq y_2 \geq \cdots \geq y_N\}. \]
Stochastic stability

This system is stable if

\[
\gamma_1 < 0, \ldots, \gamma_1 + \cdots + \gamma_{N-1} < 0, \quad \gamma_1 + \cdots + \gamma_N = 0.
\]

Most basic example ("Atlas" Configuration):

\[
\gamma = g > 0, \quad \gamma_1 = \cdots = \gamma_{N-1} = -g, \quad \gamma_N = (N-1)g.
\]

- Under this condition, we have the SLLN for local times

\[
\lim_{T \to \infty} \frac{1}{2T} \Lambda^{(k,k+1)}(T) = \lambda_k := - (\gamma_1 + \cdots + \gamma_k) > 0, \quad \text{a.s.}
\]

for \( k = 1, \ldots, N-1 \).
Under this stability condition, the vector process of gaps

\[ G^{(k,k+1)}(\cdot) = Y_k(\cdot) - Y_{k+1}(\cdot) = \log \left( \frac{\mu_k(\cdot)}{\mu_{k+1}(\cdot)} \right), \quad k = 1, \ldots, N-1 \]

converges in distribution to a random vector \((G_1, \ldots, G_{N-1})\) with exponential marginals

\[ \mathbb{P}(G_k > u) = e^{-ur_k}, \quad u > 0 \quad \text{where} \quad r_k := \frac{4\lambda_k}{\sigma_k^2 + \sigma_{k+1}^2} = \frac{1}{\mathbb{E}(G_k)}. \]

The components \(G_1, \ldots, G_{N-1}\) of this vector are \textit{independent}, if the variances are either all the same or grow linearly by rank:

\[ \sigma_2^2 - \sigma_1^2 = \cdots = \sigma_N^2 - \sigma_{N-1}^2 \geq 0. \]
Rank-based variances, U.S., 1964–2012 (smoothed)
Estimated $\gamma_k$, U.S. Equity market, 1964-2012.
Size-based approaches

The study of rank-based models such as first-order models leads to local times. Can they be avoided?

Consider a simplistic size-based model is of the form

\[ dY_i(t) = G(Y_i(t)) \, dt + S(Y_i(t)) \, dW_i(t) \]

for suitable \( G, S \). This is not realistic, since there may be no stability in \((Y_1(\cdot), \ldots, Y_N(\cdot))\).

- Refine by choosing \( G \) and \( S \) so that \((Y_1(\cdot), \ldots, Y_N(\cdot))\) has an invariant distribution, then set \( X_i(t) = M(t) \exp Y_i(t) \) for some market growth factor \( M(t) \), independent of \( Y_i(t) \).
- Alternatively, use the weight-based variant

\[ dY_i(t) = \tilde{G}(\mu_i(t)) \, dt + \tilde{S}(\mu_i(t)) \, dW_i(t). \]
Example: volatility-stabilized model

With $\alpha \geq 0$, the weight-based model (Fernholz & Karatzas 2004)

$$dY_i(t) = \frac{\alpha}{2\mu_i(t)} dt + \frac{1}{\sqrt{\mu_i(t)}} dW_i(t)$$

is known as a volatility-stabilized model.

- The coordinates can be expressed as time-changed square Bessel processes of dimension $2(1 + \alpha)$.
- Coherence holds for $\alpha > 0$ but fails for $\alpha = 0$.
- The invariant distribution of market weights is Dirichlet (Pal 2011).
- Many other properties are known (e.g. Goia 2009, Shkolnikov 2011).
Example: logarithmic potential

Stabilization from the previous model follows from unbounded volatility (and growth rate, for $\alpha > 0$). Consider instead the size-based model

$$dY_i(t) = \left( -g + \frac{\sigma^2}{2Y_i(t)} \right) dt + \sigma \exp(-\alpha Y_i(t)) \, dW_i(t),$$

for $g > 0, \sigma > 0, \alpha \geq 0$.

- Coordinates are now independent.
- With $\lambda = \frac{2g}{\sigma^2}$, the stable density $G(\cdot)$ on $(0, \infty)$ is

$$G(x) = \begin{cases} C \exp\left(2\alpha x - \frac{\lambda}{2\alpha} e^{2\alpha x} + \text{Ei}(2\alpha x)\right), & \text{if } \alpha > 0 \\ \lambda^2 x e^{-\lambda x}, & \text{if } \alpha = 0. \end{cases}$$

- Very good agreement with observed data.
A jump model

A potential barrier in the form of an unbounded growth rate may seem unrealistic (similar to Atlas stock).

Consider instead a jump model:

\[ dY_i(t) = -\gamma \, dt + \sigma \, dW_i(t) \]

if \( Y_i(t) > 0 \);

when \( Y_i(t) = 0 \), it is “kicked back” into \((0, \infty)\) at random location \( \eta \) chosen (independently) according to the distribution

\[ F(x) = \mathbb{P}(\eta \leq x), \quad 0 < x < \infty. \]

Interpretation:

- If the relative stock capitalization is too small, the stock exits the market (bankruptcy).
- It is immediately replaced by a new stock (IPO).
- This gives a constant number of stocks for convenience; a reinterpretation as a Poisson point process would allow a variable number of stocks.
Stability in the jump model

As described in Gihman & Skorokhod (1972), the system has a stable distribution $G(\cdot)$ if

$$
\int_0^\infty e^{-2\gamma x} \left( \int_0^x e^{2\gamma y} (1 - F(y)) \, dy \right) \, dx < \infty.
$$

Question: how can we ensure that $F \equiv G$?

Assume the “kicks” are absolutely continuous w.r.t. Lebesgue measure. Then

$$1 - F(x) = \int_x^\infty f(u) \, du$$

for some pdf $f(\cdot)$ on $(0, \infty)$. 
It develops that

\[ f(x) = \frac{N^f(x)}{\int_0^\infty N^f(u) \, du}, \]

where

\[ N^f(x) := e^{-2\gamma x} \int_0^x e^{2\gamma y} \left( \int_y^\infty f(u) \, du \right) \, dy. \]

If \( f(\cdot) \) is \( C^2 \), it must then solve the ODE

\[ f''(x) + 2\gamma f'(x) + f'(0^+) f(x) = 0, \quad x \in (0, \infty). \]

A solution (not known to be unique!) is given by

\[ f(x) = \left( \frac{\gamma}{\sigma^2} \right)^2 x e^{-\gamma x/\sigma^2}, \]

leading to a Gamma distribution with parameter 2, bearing more than a passing resemblance to the analogous distribution for the logarithmic potential with constant variance (\( \alpha = 0 \)).
Summary

- Macroscopic models should be consistent with observations of stability and growth rates.
- All the models considered here have a stabilization mechanism at the “bottom” (Atlas stock, logarithmic potential, bankruptcy/IPO).
- Stabilization may also occur at the “top”, e.g. antitrust.
- Many open questions and generalizations, e.g.:
  - Understand ranks and local time properties for non-rank-based models
  - Study properties of canonical portfolios, e.g. equal-weight, universal, in these models
  - Better linkage of common features between these models
- Thank you!
A few references


