Hedging market risk in optimal liquidation

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What is the OFR?

▶ Committee to establish the National Institute of Finance (NIF)
▶ Established by Dodd-Frank ’10 to support the Financial Stability Oversight Council (a senior risk management committee for the financial system), its member agencies, and the public:
  ▶ monitoring and evaluating potential threats to financial stability
  ▶ conducting and sponsoring research related to financial stability
  ▶ promoting best practices in risk management
  ▶ helping the Council and its members develop and evaluate policies for achieving financial stability
  ▶ addressing the gaps in financial data and helping to fill them, and promoting data integrity, accuracy, and transparency for the benefit of market participants, regulators, and research communities
▶ OFR = Research and Analysis Center + Data Center
▶ RAC = Current Analysis + Policy Studies + Fundamental Research

J.P. Fouque: member of NIF, current member of OFR’s FRAC
What does a mathematician do at OFR?

- OFR has an interdisciplinary research staff, but most are PhD economists. A mathematician has a complementary skill set.
- Current analysis: (i) the OFR’s Financial Stress Index, (ii) natural language processing monitoring product, (iii) AR: effect of interest rate shock; ETFs and financial stability; volatility paradox.
- Policy studies: (i) FSOC AR: developments in equities, commodities, MF, HF, PF, ETFs, (ii) effect of countercyclical capital buffer.
- Fundamental research:
  - Hedging Market Risk in Optimal Liquidation
  - On the Optimal Wealth Process...
  - Trust and Stability in a Financial Network
  - Crisis Greeks
  - Asset Managers with Multiple Heterogeneous Clients
  - Contingent Claim Analysis
  - Polysplines for Portfolio Payoffs
Hedging market risk in optimal liquidation

- Motivation and background
- Market model and terminal portfolio value
- Solution for CARA utility
- Analysis and plots of optimal strategies
- Broker-dealer’s minimum spread
Motivation

Standard models assume markets are perfectly liquid.

Investors are often liquidity demanders.

Instantaneous liquidity often not available or expensive:
  ▶ Find a dark pool (recent talk of increased regulation)
  ▶ Pay a broker-dealer’s block/capital markets desk
  ▶ Standard solution: break it up and liquidate over time

Liquidating over time involves market risk.

Take symmetric position in correlated but relatively liquid asset.

Is this behavior “optimal”? What is the hedge?
Setup

Investor must liquidate a large long position in a primary asset over [0, T].

Investor is liquidity demander; no inside information.

Proceeds deposited into a riskfree money market account.

Investor trades between money market and liquid proxy.

Investor maximizes expected terminal utility.

Preview of results:
- Almgren-Chriss type model with liquid proxy.
- Optimal strategies deterministic, found explicitly.
- Hedge for market risk effectively makes for more aggressive liquidation.
- Indifference price (spread) for broker-dealer trading as principal.
- Always better to find and trade in liquid proxy.
Background

Numerous studies (Kraus & Stoll ('72), Holthausen et al ('87,'90), Keim & Madhavan ('95), Almgren et al ('05), Frino et al ('06), etc.)

- Temporary price impact
- Permanent price impact

Many microstructure models (Kyle ('85), Easley & O’Hara ('87), etc.) attempt to explain these endogenously.

A separate line takes price impact effects as exogenous and then derives optimal strategies.

Most popular is Almgren-Chriss ('99,'00,'03) model:
- Incorporates temporary and permanent price impacts.
- Mathematically tractable.
- Widely used in academia and practice.
Almgren-Chriss type market environment

Over the horizon \([0, T]\), the market consists of

- Riskless money market that pays no interest;
- Proxy with price \(S_t\) given by Bachelier model with drift:

\[
S_t = S_0 + \mu t + \sigma W_t
\]

- Primary asset with price \(S^I_t\) following simple Almgren-Chriss model:

\[
S^I_t = S^I_0 + \mu_I t + \sigma_I W^I_t + \gamma (\eta_t - \eta_0) - \theta \xi_t
\]

- \(\eta_t\), number of shares at time \(t\) (absolutely continuous),
- \(\xi_t\), speed of liquidation, i.e. \(\eta_t = \eta_0 - \int_0^t \xi_u du\) (uniformly bounded),
- \(\gamma \geq 0\), coefficient of permanent price impact,
- \(\theta > 0\), coefficient of temporary price impact,
- \(d \langle W, W^I \rangle_t = \rho \, dt\), \(\rho \in [0, 1)\).
Portfolio value

1. Initial money market account value is zero.
2. Liquidation: investor sells $\xi_t \, dt$ shares at time $t$ for price $S^l_t$
   - Value at time $t$ is $\int_0^t \xi_s S^l_s \, ds$.
3. Form portfolio with money market and proxy.

Value at terminal time $T$, using $\eta_T = 0$:

$$X_T^{\pi, \xi} = x_0 + \mu I \int_0^T \eta_s \, ds + \sigma I \int_0^T \eta_s \, dW^I_s - \theta \int_0^T \xi^2_s \, ds$$
$$+ \mu \int_0^T \pi_s \, ds + \sigma \int_0^T \pi_s \, dW_s,$$

where
- $\pi_t$, number of shares in proxy asset at time $t$ (uniformly bounded),
- $x_0 := S^l_0 \eta_0 - \frac{\gamma}{2} \eta^2_0$.

Regularity
Solution for exponential utility

The investor’s objective is to find the policy \((\pi, \xi) \in \mathcal{A}\) that solves

\[
\sup_{(\pi, \xi) \in \mathcal{A}} \mathbb{E}[-\exp(\alpha X_{T}^{\pi,\xi})], \quad \alpha > 0.
\]

Conjecture: optimal strategy is deterministic

Intuition: optimal strategies for CARA investor are deterministic in:
- standard Merton model for optimal investment,
- “pure” liquidation model ([Schied, Schöneborn & Tehranchi (’10)]).
Theorem

Let the positive constant $\kappa_\rho$ be defined by

$$\kappa_\rho := \sqrt{\frac{\alpha \sigma_I^2 (1 - \rho^2)}{2\theta}}.$$ 

Then, the investor’s unique optimal policy is the deterministic strategy $(\pi^*, \xi^*)$ given by

$$\pi^*_t = \frac{1}{\alpha} \frac{\mu}{\sigma^2} - \rho \frac{\sigma_I}{\sigma} \eta^*_t,$$

and

$$\xi^*_t = \kappa_\rho \eta_0 \frac{\cosh(\kappa_\rho (T - t))}{\sinh(\kappa_\rho T)} + \left( \frac{\mu}{\rho \sigma} - \frac{\mu_I}{\sigma_I} \right) \frac{e^{\kappa_\rho (T-t)} - e^{\kappa_\rho t}}{\sqrt{2\alpha \theta (1 - \rho^2)(e^{\kappa_\rho T} + 1)}},$$

where

$$\eta^*_t = \eta_0 \frac{\sinh(\kappa_\rho (T - t))}{\sinh(\kappa_\rho T)} - \left( \frac{\mu}{\rho \sigma} - \frac{\mu_I}{\sigma_I} \right) \frac{(e^{\kappa_\rho (T-t)} - 1)(e^{\kappa_\rho t} - 1)}{\alpha (1 - \rho^2) \sigma_I (e^{\kappa_\rho T} + 1)}.$$
Broker-dealer’s indifference price

Proposition

The investor’s indifference price $h(\eta_0, 0)$ at time $t = 0$ is given by

$$h(\eta_0, 0) = \frac{\gamma}{2} \eta_0^2 + \theta \int_0^T (\xi_t^*)^2 \, dt + \frac{\alpha}{2} (1 - \rho^2) \sigma_i^2 \int_0^T (\eta_t^*)^2 \, dt$$

$$- \left( S_0 \eta_0 + \frac{1}{2\alpha} \frac{\mu^2}{\sigma^2} T + \mu \eta_0 (1 - \rho) \int_0^T \eta_t^* \, dt \right).$$
The no-drift market, $\mu = \mu_I = 0$

**Figure:** Model parameters:

$\alpha = 10, \sigma = 0.03 = \sigma_I = 0.03, \gamma = 0.3, \theta = 0.05, \eta_0 = 10, S_I^0 = 1.5$. Median 6-mo correlation S&P 500: 0.55, CBOE Avg Imp Imp Corr Index (Jan 14): 0.55.
The no-drift market, $\mu = \mu_I = 0$

**Figure:** Model parameters:
\[\alpha = 10, \sigma = 0.03 = \sigma_I = 0.03, \gamma = 0.3, \theta = 0.05, \eta_0 = 10, S_I^0 = 1.5.\] Median 6-mo correlation S&P 500: 0.55, CBOE Avg Imp Corr Index (Jan 14): 0.55.
The no-drift market, \( \mu = \mu_I = 0 \)

![Diagram showing position in primary asset over time with different correlation values.]

**Figure:** Model parameters:

\[ \alpha = 10, \sigma = 0.03 = \sigma_I = 0.03, \gamma = 0.3, \theta = 0.05, \eta_0 = 10, S_I^0 = 1.5. \]

Median 6-mo correlation S&P 500: 0.55, CBOE Avg Imp Corr Index (Jan 14): 0.55.
Proposition

The following assertions hold in the no-drift market:

i) We have

\[ \lim_{\rho \uparrow 1} \xi_t^* = \frac{\eta_0}{T}, \quad 0 \leq t \leq T. \]

ii) Holding all other parameters fixed, we have that

\[ \xi_t^*(\alpha, \rho) = \xi_t^*(\hat{\alpha}, \hat{\rho}) \quad \text{and} \quad \eta_t^*(\alpha, \rho) = \eta_t^*(\hat{\alpha}, \hat{\rho}) \]

for all \( t \in [0, T] \), if and only if

\[ \alpha(1 - \rho^2) = \hat{\alpha}(1 - \hat{\rho}^2). \]

iii) Holding all other parameters fixed, we have that

\[ \xi_t^*(\theta, \rho) = \xi_t^*(\hat{\theta}, \hat{\rho}) \quad \text{and} \quad \eta_t^*(\theta, \rho) = \eta_t^*(\hat{\theta}, \hat{\rho}) \]

for all \( t \in [0, T] \), if and only if

\[ \frac{1 - \rho^2}{\theta} = \frac{1 - \hat{\rho}^2}{\hat{\theta}}. \]
Proposition
The following assertions hold in the no-drift market:

i) The investor’s value function at initial time is

\[ v(0, x_0, \eta_0) = -\exp \left( -\alpha x_0 + \alpha \theta \kappa \rho \coth(\kappa \rho T) \eta_0^2 \right), \]

where \( x_0 = S_0^l \eta_0 - \frac{\gamma}{2} \eta_0^2 \).

ii) The investor’s value function is increasing in the correlation \( \rho \).

iii) The investor’s indifference price at initial time \( t = 0 \) is given by

\[ h(\eta_0, 0) = \left( \frac{\gamma}{2} + \theta \kappa \rho \coth(\kappa \rho T) \right) \eta_0^2 - S_0^l \eta_0. \]
Conclusions and future directions

- Almgren-Chriss type model with liquid proxy.
- Optimal strategies deterministic, found explicitly.
- Hedge for market risk effectively makes for more aggressive liquidation.
- Indifference price for broker-dealer trading as principal.
- Always better to find and trade in liquid proxy.

Future directions

- Endogenous liquidation time. Does presence of proxy increase liquidation horizon? Recent work by Bechler & Ludkovski ('14).
- More general models and risk criteria. Several large traders.
Thank you!
Regularity

Regularity: optimal strategies for reasonable criteria exist and are well-behaved.

Formed in terms of expected revenues in no-drift market.

Three popular notions. Absence of

- Price manipulation (Huberman & Stanzl ('04))
- Transaction-triggered price manipulation (Alfonsi et al ('12))
- Negative expected execution costs (Klöck et al ('12))

Our model satisfies all three regularity conditions.

Expected terminal wealth (no-drift): $\mathbb{E}[X_T^\pi,\xi] = x_0 - \theta \int_0^T \xi_t^2 dt$.

Jensen’s inequality implies $\xi^*_t = \frac{\eta_0}{T}$.

So-called VWAP (Volume-Weighted Average Price) strategy:

$X_T^{\pi^*,\xi^*} = \eta_0 \left( \frac{1}{T} \int_0^T S_t^l dt \right)$. 
Strategy and sketch of proof

1. Show

\[
\sup_{(\pi, \xi) \in A} \mathbb{E}[- \exp(\alpha X_{T}^{\pi, \xi})] = \sup_{(\pi, \xi) \in A_{\text{det}}} \mathbb{E}[- \exp(\alpha X_{T}^{\pi, \xi})].
\]

2. Calculus of variations to find unique maximizer \((\pi^*, \xi^*)\) over \(A_{\text{det}}\).

3. Uniqueness follows from strict concavity of \((\pi, \xi) \mapsto \mathbb{E}[u(X_{T}^{\pi, \xi})]\).

Sketch of proof: Write, for general \((\pi, \xi) \in A\),

\[
\mathbb{E}[u(X_{T}^{\pi, \xi})] = -e^{-\alpha x_{0}} \mathbb{E} \left[ e^{Y_{T}^{\pi, \xi} + f(\pi, \xi)} \right],
\]

where

\[
Y_{T}^{\pi, \xi} = -\alpha \left( \sigma \int_{0}^{T} \pi_{t} dW_{t} + \int_{0}^{T} \eta_{t} \sigma [\rho dW_{t} + \sqrt{1 - \rho^{2}} dW_{t}^{\perp}] \right),
\]

\[
f(\pi, \xi) = -\alpha \left( \mu \int_{0}^{T} \pi_{t} dt + \mu_{1} \int_{0}^{T} \eta_{t} dt - \theta \int_{0}^{T} \xi_{t}^{2} dt \right).
\]
Then, for general \((\pi, \xi) \in \mathcal{A}\),

\[
\mathbb{E}[u(X_T^{\pi, \xi})] = -e^{-\alpha x_0} \mathbb{E} \left[ e^{Y_T^{\pi, \xi} + f(\pi, \xi)} \right]
\]

\[
= -e^{-\alpha x_0} \mathbb{E} \left[ e^{Y_T^{\pi, \xi} - \frac{1}{2} \langle Y^{\pi, \xi} \rangle_T} \cdot e^{\frac{1}{2} \langle Y^{\pi, \xi} \rangle_T + f(\pi, \xi)} \right]
\]

Radon-Nikodym derivative?
Then, for general \((\pi, \xi) \in \mathcal{A}\),

\[
\mathbb{E}[u(X_{T}^{\pi, \xi})] = -e^{-\alpha x_{0}} \mathbb{E} \left[ e^{Y_{T}^{\pi, \xi}} + f(\pi, \xi) \right]
\]

\[
= -e^{-\alpha x_{0}} \mathbb{E} \left[ e^{Y_{T}^{\pi, \xi} - \frac{1}{2} \langle Y_{T}^{\pi, \xi} \rangle_{T}} e^{\frac{1}{2} \langle Y_{T}^{\pi, \xi} \rangle_{T}} + f(\pi, \xi) \right]
\]

\[
= -e^{-\alpha x_{0}} \mathbb{E}^{\pi, \xi} \left[ e^{\frac{1}{2} \langle Y^{\pi, \xi} \rangle_{T} + f(\pi, \xi)} \right]
\]

\[
\leq -e^{-\varepsilon} e^{-\alpha x_{0}} \mathbb{E}^{\pi, \xi} \left[ e^{\frac{1}{2} \langle Y^{\pi, \xi} \rangle_{T} + f(\pi^{e}, \xi^{e})} \right]
\]

\[
= -e^{-\varepsilon} e^{-\alpha x_{0} + \frac{1}{2} \langle Y^{\pi^{e}, \xi^{e}} \rangle_{T} + f(\pi^{e}, \xi^{e})}
\]

\[
= e^{-\varepsilon} \mathbb{E}[u(X_{T}^{\pi^{e}, \xi^{e}})].
\]
Maximizing over \((\pi, \xi) \in A_{\text{det}}\) is equivalent to minimizing
\[
\int_0^T F(t, y(t), y'(t))\,dt, \quad y(t) = (\pi_t, \eta_t),
\]
for some specific \(F\), over curves with \(y(0) = (0, \eta_0)\) and \(y(T) = (\pi, 0)\).

Euler-Lagrange and strict convexity imply that the unique solution is the solution to the second-order ODE
\[
2\theta \ddot{\eta}_t - (1 - \rho^2)\alpha \sigma_I^2 \eta_t = \frac{\mu}{\sigma} \rho \sigma_I - \mu_I,
\]
with boundary conditions
\[
\eta_0 = \eta_0 > 0, \quad \eta_T = 0.
\]
Heuristic arguments suggest that the value function, defined by

\[ v(T - t, x, \eta) := \sup_{(\pi, \xi) \in A} \mathbb{E}[u(X_T^{\pi, \xi}) | X_t^{\pi, \xi} = x, \eta_t^{\pi} = \eta], \]

satisfies the Hamilton-Jacobi-Bellman equation,

\[ v_t = \frac{1}{2} \sigma^2 \eta^2 v_{xx} + \mu \eta v_x \]

\[ + \sup_{(\pi, \xi) \in A} \left[ (\mu v_x + \rho \sigma \eta v_{xx}) \pi + \frac{1}{2} \sigma^2 v_{xx} \pi^2 - v_{\eta} \xi - \theta v_x \xi^2 \right] = 0, \]

subject to the terminal condition

\[ \lim_{t \uparrow T} v(t, x, \eta) = \begin{cases} -e^{-\alpha x}, & \eta = 0 \\ -\infty, & \text{otherwise.} \end{cases} \]

One shows that the our value function is a smooth solution to the HJB.