Optimal Execution with Dynamic Order Flow Imbalance

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Outline

1. Order Flow
2. Dynamic Order Flow Model
3. Two-Step Approximation
4. Numerical Illustrations
Tale of Two Costs

When executing a large order in an electronic market, there are two primary concerns:

1. Price Impact
   - Spatial effect: consume liquidity in terms of standing limit orders
   - “Walking through the book” – receive worse price than current best bid/ask
Tale of Two Costs

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1. **Price Impact**
   - **Spatial** effect: consume liquidity in terms of standing limit orders
   - “Walking through the book” – receive worse price than current best bid/ask

2. **Information Leakage**
   - **Temporal** effect: executed trade appears on the order tape
   - Other traders react and adjust their behavior (eg “front-running”) – will receive worse price in the future
Price Impact
Example: A large market SELL order,
- The order will first deplete standing limit buy orders at bid price and then move to next best level, lowering execution price
- Price impact depends on the shape of the LOB
- A well studied problem
  - Flat (constant depth) LOB → linear price impact
  - Flat LOB → quadratic instantaneous execution cost

![Price Impact Diagram 1](image1.png)

![Price Impact Diagram 2](image2.png)
Information Leakage

- Longer term impact causing others to behave differently (i.e., permanent market impact)
- Once other traders are aware of somebody selling, they will take advantage
- Adverse selection, predatory trading, etc.
- Algorithm should do the following:
  - Attempt to hide overall intentions
  - Consider the state of the liquidity provision process
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- Adverse selection, predatory trading, etc.
- Algorithm should do the following:
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  - Consider the state of the liquidity provision process
- Information cost requires looking at time series of orders
Order Flow

- Participants fear that other traders are better informed
  - When order flow is toxic market makers provide less liquidity
- Toxicity is about the ratio of noise/informed traders, aka flow imbalance (NOT the same as static/volume LOB order imbalance)
- Not directly observable, and not obvious how to measure


- VPIN has a complicated way of classifying trades as buys/sells. Not suitable for LOB data
- Claim: MMs widen spreads in response to toxic flow

FinMath: Cartea, Jaimungal, Ricci (2011)

- Short term price drift driven by market order arrivals
- Result: MMs shift order placement when market order flow is one-sided
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Waves of Coca Cola

Figure: Intra-day Coca-Cola Stock Price on NYSE on July 19, 2012
Direction

- **ELO12**: Flow imbalance, info leakage and optimal horizon
  - Static, one-period model
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- Aim: Incorporate concepts in a dynamic optimal execution framework
  - Continuous trading rates (Almgren-Chriss,...)
  - Assume a parametric form for inventory risk (A-C, Gatheral-Schied)
  - No fill risk (in contrast to Cartea & Jaimungal, Gueant et al, Bayraktar-L, ...)
  - Treat only market orders (in the future: hybrid models like in Guilbaud & Pham, Carmona & Webster, ...)
  - (Order book resilience (Obizhaeva & Wang, Alfonsi et al))
  - (Empirical evidence: Farmer, Bouchaud, ...)

Bechler
Execution & Order Flow
Contributions

- Introduce order flow imbalance as a state variable
- Suggest a simple mechanism for informational costs (inspired by ELO12) – temporal impact beyond usual spatial impact
- Endogenize the execution horizon $T$
- Derive closed-form approximate strategies for the resulting optimization problem
Execution Model

- Inventory $x_t$: liquidate $x_0$
- $t \mapsto x_t$ is absolutely continuous; trading rate is $\alpha_t$
- $dx_t = -\alpha_t dt$
- Unaffected price $S_t$: martingale
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- Unaffected price $S_t$: martingale
- Order flow imbalance $Y_t$
  - $Y_t > 0$: buyers-market; $Y_t < 0$: sellers market
- Main desired features:
  - Mean-reverting to zero
  - Stationary in long-run
  - Affected by execution algorithms
Unaffected order flow: \( dY_t^0 = -\beta Y_t^0 dt + \sigma dW_t \)

With execution:

\[
dY_t = (-\beta Y_t - \phi(\alpha_t)) dt + \sigma dW_t
\]

i.e. \( Y_t = Y_t^0 + \int_0^t e^{-\beta(t-s)} \alpha_s ds \)

Typical cases:

- \( \phi(\alpha) = \phi_t \) (deterministic information cost)
- \( \phi(\alpha) = \eta \alpha \) (linear in trading rate)

Assume that flow is independent of price (empirical relationship is not clear) - more on this later
Optimization Problem

- **Objective:** \( v(x, y) := \inf_{\alpha \in A} \mathbb{E}_{x,y} \left[ \int_0^{T_0} g(\alpha_s) + \lambda(x_s) + \kappa Y_s^2 \, ds \right] \)

- **Realized horizon** \( T_0 := \inf \{ t : x_t^\alpha = 0 \} \) – endogenous to the strategy \( \alpha \)
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- \( \lambda(x) \): inventory risk
- \( \lambda(x) = cx^2 \) (Almgren-Chriss criterion) / \( \lambda(x) = cx \) (similar to Gatheral-Schied)
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- \( \kappa Y^2 \): information cost
- Unbalanced order flow: Higher liquidity costs
HJB Equation

0 = −β Yν_ν_Y + \frac{1}{2} \sigma^2 ν_ν_ν_Y + \inf_{\alpha \geq 0} \{g(\alpha) - \alpha ν_X - \phi(\alpha) ν_Y\} + \kappa Y^2 + \lambda(x)

- Finite-fuel boundary condition: ν(0, y) = 0 for all y
- **Nonlinear parabolic PDE**
- Hard to understand the structure
- Positivity constraint on α is challenging
HJB Equation

\[ 0 = -\beta Y v_Y + \frac{1}{2} \sigma^2 v_{YY} + \inf_{\alpha \geq 0} \{ g(\alpha) - \alpha v_x - \phi(\alpha) v_Y \} + \kappa Y^2 + \lambda(x) \]

Finite-fuel boundary condition: \( v(0, y) = 0 \) for all \( y \)

Nonlinear parabolic PDE

Hard to understand the structure

Positivity constraint on \( \alpha \) is challenging

To gain insights: build approximating problems by

(i) solving the fixed-horizon problem

(ii) optimizing over \( T \)
Fixed Horizon Problem

\[ u(T, x, y) = \inf_{(\alpha_t) \in A(T,x)} \mathbb{E}_{x,y} \left[ \int_0^T \alpha_s^2 + \lambda(x_s^\alpha) + \kappa y_s^2 \, ds \right] \]

HJB equation becomes

\[ u_T = \frac{1}{2} \sigma^2 u_{yy} + \kappa y^2 + \lambda(x) - \beta y u_y + \inf_{\alpha} \left\{ \alpha^2 - \alpha u_x - \eta \alpha u_y \right\} \] (1)

Singular boundary condition: \( \lim_{T \downarrow 0} u(T, x, y) = \infty \) if \( x \neq 0 \)

Proposition

The solution of (1) has the form

\[ u(T, x, y) = x^2 A(T) + y^2 B(T) + xyC(T) + D(T), \] (2)

where \( A, B, C, D \) solve a matrix Riccati ordinary differential equation.

Note: Riccati equations parameterized in terms of time-to-maturity \( \tau \)
Execution Speed

- The corresponding optimal rate of liquidation is

$$\alpha_t^D = \frac{x_t(2A(\tau) + \eta C(\tau)) + Y_t(C(\tau) + 2\eta B(\tau))}{2}.$$  

- Execution rate is linear in $x_t$ and in $Y_t$ (generalizes Almgren-Chriss)

- The Proposition only treats the unconstrained case $\alpha \in \mathbb{R}$: if $T$ is large relative to $x_0$ or $Y_t$ is negative enough then $\alpha^D < 0$

- As $t \to T$, the dynamic trading rate stabilizes, resembling a VWAP strategy.
Myopic Strategies \( (\phi(\alpha) = \phi_t) \)

- Suppose agent myopically optimizes only against price impact:
  \[
u^M(T, x, y) := \inf_{x_t} \left( \int_0^T \dot{x}_s^2 + \lambda(x_s) ds \right) + \int_0^T \kappa E_y [Y_s^2] ds =: \mathcal{I} + \mathcal{O}\]

- If \( \lambda(x) = cx^2 \) solution is
  \[
  \begin{cases}
  x_t^{MH} = \frac{x \sinh(\sqrt{c}(T - t))}{\sinh(\sqrt{c}T)} \\
  \alpha_t^{MH} = \frac{\sqrt{cx} \cosh(\sqrt{c}(T - t))}{\sinh(\sqrt{c}T)}
  \end{cases}
  \]

- Now \( \phi_t = \eta \alpha_t^{MH} \) is the above deterministic function of \( t \rightarrow Y_t \) is Gaussian with known moments

- \( \mathcal{I}^{MH}(T, x) = \sqrt{cx^2} \coth(\sqrt{c}T) \)

- \( \mathcal{O}^{MH}(T, x, y) = \kappa \int_0^T \left( ye^{-\beta t} - \int_0^t e^{-\beta(t-s)} \eta \alpha_s^{MH} ds \right)^2 + \frac{\sigma^2}{2\beta} \left( 1 - e^{-2\beta t} \right) dt \)
Myopic Strategies \((\phi(\alpha) = \phi_t)\)

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\[
I^{MH}(T, x) = \sqrt{cx^2} \coth(\sqrt{c}T)
\]

\[
O^{MH}(T, x, y) = \kappa \int_0^T \left( ye^{-\beta t} - \int_0^t e^{-\beta(t-s)} \eta\alpha_s^{MH} \, ds \right)^2 + \frac{\sigma^2}{2\beta} \left( 1 - e^{-2\beta t} \right) dt
\]

- Similarly have closed-form expressions for cases \(\lambda(x) = cx\) (Quadratic) and \(\lambda(x) = 0\) (VWAP)

- Next step: Take existing closed-form expressions \(u^M\) and optimize over \(T\)
Optimizing the Horizon

- \( T^* = \arg \min_T u(T, x, y) \)
- Lemma: \( T^* \in (0, \infty) \) (closed-form for \( T \mapsto u(T, x, y) \))
- Open-loop (static): find \( T^* \) at the outset and implement \( \alpha_t(T^*(x, y), x_t, Y_t) \)

Realized horizon \( T_0(x, y) \) becomes random

Dynamically recomputing \( T^* \) - adapt to changing \( Y_t \) without the indefinite horizon finite-fuel problem

Next up: we show these are in fact good approximations!
Optimizing the Horizon

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- Open-loop (static): find \( T^* \) at the outset and implement \( \alpha_t(T^*(x, y), x_t, Y_t) \)
- Closed-loop (dynamic): continuously recompute \( T^* \):
  \[
  \tilde{\alpha}_t^M(x, y) := \alpha^M(T^*(x_t, Y_t), x_t, Y_t)
  \]
- Realized horizon \( T_0(x, y) \) becomes random
- Dynamically recomputing \( T^* \) - adapt to changing \( Y_t \) without the indefinite horizon finite-fuel problem

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Comparison of costs

<table>
<thead>
<tr>
<th></th>
<th>( \nu )</th>
<th>( \tilde{\nu}^D )</th>
<th>( \tilde{\nu}^{ML} )</th>
<th>( u^D )</th>
<th>( u^{ML} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathbb{E}[J(\alpha)] )</td>
<td>4.257</td>
<td>4.264</td>
<td>4.317</td>
<td>4.483</td>
<td>4.547</td>
</tr>
<tr>
<td>( SD(J(\alpha)) )</td>
<td>1.50</td>
<td>1.45</td>
<td>1.39</td>
<td>1.77</td>
<td>1.84</td>
</tr>
<tr>
<td>( \mathbb{E}[T_0] )</td>
<td>3.87</td>
<td>3.70</td>
<td>3.48</td>
<td>3.43</td>
<td>3.43</td>
</tr>
</tbody>
</table>

Legend:
- \( \nu \): directly from the HJB pde (fully numerical)
- \( \tilde{\nu} \): closed-loop optimization of \( T^* \); \( T_0 \) is random
- \( u(T^*, x, y) \): static optimization \( T_0 = T^* \)
- \( u_t^{ML} = x / T \) (VWAP)
- \( u_t^D \) based on Proposition 1 (matrix \textbf{Riccati} equations)
Figure: Comparison of trading rates ($\alpha_t$) for each of six strategies given the shown simulated path of ($Y_t^0$) (The realized ($Y_t$) depends on the strategy chosen).
Figure: Top: 200 simulated trajectories from strategy $\tilde{\alpha}_t^D$. Highlighted are three trajectories resulting from different $Y_t$-paths. Bottom: Corresponding realizations of $t \mapsto Y_t$. 
More on Static $T^*$ (Optimal Execution Horizon by ELO12)

- ELO12 (essentially) considered $\min_{T \geq 0} \{\mathbb{E}_{x, y} \left[ |Y_T^\alpha| \right] + c\sqrt{T}\}$
- myopic trading based on constant participation strategy $\alpha_t = x/T$; explicit timing costs (rather than inventory risk)
- Only terminal information costs $|Y_T|
- Discuss the statistically optimized $T^*$ above
- Our setup is effectively a dynamic extension of the above one-period model
Work in Progress

- **Empirical** measurement of order flow and its stylized features
- At what **time-scale** is order flow imbalance to be measured?
- Perhaps information leakage $\phi(\alpha)$ depends on $Y_t$?
- Consider **correlated** $S_t$ and $Y_t$
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Thank You!
Empirical Executed Order Flow

Figure: The EWMA order flow imbalance metric for Teva Pharmaceutical (ticker: TEVA) for a single day 5/3/2011.
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