Commodities, Derivatives on Futures, and Multiscale Volatility Models

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joint work with Jean-Pierre Fouque$^2$ and Yuri F. Saporito$^2$

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September 27, 2014
Long term cooperation agreement with PETROBRAS
Prologue

- Long term cooperation agreement with PETROBRAS
- Cooperation with BMF & BOVESPA
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- Math Finance professional Master’s program
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J-P.
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- J-P.
United States Wellhead Prices

United States Natural Gas Wellhead Prices (monthly)

Time

Log Price

Seasonality in Term Structure

Henry Hub Nymex Futures 15-AUG-2005

Futures Log Price vs. Num Months to Expiration
Seasonality in Storage

Weekly Lower 48 States Natural Gas Working Underground Storage

Billion Cubic Feet

Source: U.S. Energy Information Administration
Commodities comprise over 23% of the traded assets in the market;
The spot commodity is generally perishable.
Every futures contract is born with an expiration date.
Every futures investment really depends upon the futures term structure.
Mean reversion is an important feature of commodity prices.
Classical Black model
Classical Black model

One and Two-Factor Models: Schwartz, Gibson - Schwartz, Schwartz - Smith
Classical Black model
One and Two-Factor Models: Schwartz, Gibson - Schwartz, Schwartz - Smith
H. Geman: *Commodities and Commodity Derivatives: Energy, Metals and Agricultural*
Intro

Literature

- Classical Black model
- One and Two-Factor Models: Schwartz, Gibson - Schwartz, Schwartz - Smith
- H. Geman: *Commodities and Commodity Derivatives: Energy, Metals and Agriculturals*
- R. Carmona and M. Coulon: *A Survey of Commodity Markets and Structural Models for Electricity Prices*
Classical Black model

One and Two-Factor Models: Schwartz, Gibson - Schwartz, Schwartz - Smith

H. Geman: *Commodities and Commodity Derivatives: Energy, Metals and Agricultural*

R. Carmona and M. Coulon: *A Survey of Commodity Markets and Structural Models for Electricity Prices*

Huge literature
Motivation

- The simplicity of the model of Black (1976) is not sufficient to provide a good understanding of the modern futures financial market.
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- Many models have been proposed (e.g. Gibson Schwartz, Schwartz Smith, HJM, etc). However, the pricing of option on futures may be extremely complicated and requires long numerical simulations.
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- We seek approximate solutions in the context of multiscale analysis proposed by Fouque, Papanicolaou, Sircar, Solna [1].
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- Many models have been proposed (e.g. Gibson Schwartz, Schwartz Smith, HJM, etc). However, the pricing of option on futures may be extremely complicated and requires long numerical simulations.
- We seek approximate solutions in the context of multiscale analysis proposed by Fouque, Papanicolaou, Sircar, Solna [1].
write the SDE for the future $F_{t,T}$ with all coefficients depending only on $F_{t,T}$. This means we will need to invert the future prices of $V$ in order to write $V_t$ as a function of $F_{t,T}$;

consider the pricing partial differential equation (PDE) for a European derivative on $F_{t,T}$.

The coefficients of this PDE will depend on the time-scales of the stochastic volatility of the asset in a complicated way. Use perturbation analysis to treat such PDE by expanding the coefficients;

determine the first-order approximation of derivatives on $F_{t,T}$ as it is done in [1].
we do not rely on the Taylor expansion of the payoff function to derive the first-order approximation

our first-order correction is a substantial improvement of earlier perturbative work.

we present a simple calibration procedure of the market group parameters. The simple expression of our first-order correction is one of the reasons such calibration procedure is possible.

the essential aspect of our method is that we consider the future price $F_{t,T}$ as the variable, and not spot price $V_t$.

So, since the future price is a martingale (as opposed to the $V$), better formulas for the Greeks of the 0-order term are available.
$V_t$ denotes the asset value and it will be modeled by an exp-OU with multiscale stochastic volatility (under a risk-neutral measure $\mathbb{Q}$):

**Time Scales:** $\varepsilon \ll T \ll 1/\delta$

\[
\begin{align*}
V_t &= e^{U_t}, \\
\frac{dU_t}{dt} &= \kappa(m - U_t)dt + \eta(Y_t^\varepsilon, Z_t^\delta)dW_t^{(0)}, \\
\frac{dY_t^\varepsilon}{dt} &= \frac{1}{\varepsilon} \alpha(Y_t^\varepsilon)dt + \frac{1}{\sqrt{\varepsilon}} \beta(Y_t^\varepsilon)dW_t^{(1)}, \\
\frac{dZ_t^\delta}{dt} &= \delta c(Z_t^\delta)dt + \sqrt{\delta} g(Z_t^\delta)dW_t^{(2)}.
\end{align*}
\]
Model

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Time Scales: $\varepsilon \ll T \ll 1/\delta$

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\begin{cases}
V_t = e^{U_t}, \\
\quad dU_t = \kappa(m - U_t)dt + \eta(Y^\varepsilon_t, Z^\delta_t)dW^{(0)}_t, \\
\quad dY^\varepsilon_t = \frac{1}{\varepsilon}\alpha(Y^\varepsilon_t)dt + \frac{1}{\sqrt{\varepsilon}}\beta(Y^\varepsilon_t)dW^{(1)}_t, \\
\quad dZ^\delta_t = \delta c(Z^\delta_t)dt + \sqrt{\delta}g(Z^\delta_t)dW^{(2)}_t.
\end{cases}
\]

Correlation: $dW^{(0)}_tdW^{(i)}_t = \rho_i dt, \ i = 1, 2, \ dW^{(1)}_tdW^{(2)}_t = \rho_{12} dt$
Model

$V_t$ denotes the asset value and it will be modeled by an exp-OU with multiscale stochastic volatility (under a risk-neutral measure $\mathbb{Q}$):

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\text{Time Scales: } & \varepsilon \ll T \ll 1/\delta \\
V_t &= e^{U_t}, \\
dU_t &= \kappa (m - U_t)dt + \eta (Y_t^\varepsilon, Z_t^\delta) dW_t^{(0)}, \\
dY_t^\varepsilon &= \frac{1}{\varepsilon} \alpha (Y_t^\varepsilon) dt + \frac{1}{\sqrt{\varepsilon}} \beta (Y_t^\varepsilon) dW_t^{(1)}, \\
dZ_t^\delta &= \delta c (Z_t^\delta) dt + \sqrt{\delta} g (Z_t^\delta) dW_t^{(2)}. \\
\end{align*}
\]

Correlation: $dW_t^{(0)} dW_t^{(i)} = \rho_i dt, \ i = 1, 2, \ dW_t^{(1)} dW_t^{(2)} = \rho_{12} dt$

Future prices: $F_{t,T} = \mathbb{E}_\mathbb{Q}[V_T \mid \mathcal{F}_t], \ 0 \leq t \leq T$
Goal: to apply singular and regular perturbation techniques to price options on $F_{t,T}$. 
**First Steps**

**Goal:** to apply singular and regular perturbation techniques to price options on $F_{t,T}$.

(i) Write a SDE for $F_{t,T}$.

(ii) Expand the coefficients of the PDE for options on $F_{t,T}$.

(iii) Determine the first-order approximation for options on $F_{t,T}$. 
Remarks about $F_{t,T}$

$$
\mathbb{E}_Q[V_T \mid U_t = u, Y_t^\varepsilon = y, Z_t^\delta = z] = h^{\varepsilon,\delta}(t, u, y, z, T) = \\
h_0(t, u, z, T) + \sqrt{\varepsilon}h_{1,0}(t, u, z, T) + \sqrt{\delta}h_{0,1}(t, u, z, T) + \cdots
$$
Remarks about $F_{t,T}$

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\mathbb{E}_Q[V_T \mid U_t = u, Y_t^\varepsilon = y, Z_t^\delta = z] = h^\varepsilon,\delta(t, u, y, z, T) = \\
= h_0(t, u, z, T) + \sqrt{\varepsilon} h_{1,0}(t, u, z, T) + \sqrt{\delta} h_{0,1}(t, u, z, T) + \cdots
$$

$$
h_0(t, u, z, T) = \exp \left\{ m + (u - m)e^{-\kappa(T-t)} + \frac{\bar{\eta}^2(z)}{4\kappa} \left(1 - e^{-2\kappa(T-t)}\right) \right\},
$$

$$
\bar{\eta}^2(z) = \langle \eta^2(\cdot, z) \rangle,
$$
Remarks about $F_{t, T}$

$$E_Q[V_T \mid U_t = u, Y_t^\varepsilon = y, Z_t^\delta = z] = h^{\varepsilon, \delta}(t, u, y, z, T) =$$

$$= h_0(t, u, z, T) + \sqrt{\varepsilon}h_{1, 0}(t, u, z, T) + \sqrt{\delta}h_{0, 1}(t, u, z, T) + \cdots$$

$$h_0(t, u, z, T) = \exp \left\{ m + (u - m)e^{-\kappa(T-t)} + \frac{\bar{\eta}^2(z)}{4\kappa} \left(1 - e^{-2\kappa(T-t)}\right) \right\},$$

$$\bar{\eta}^2(z) = \langle \eta^2(\cdot, z) \rangle,$$

$$h_{1, 0}(t, u, z, T) = g(t, T)V_3(z)\frac{\partial^3 h_0}{\partial u^3}(t, u, z, T),$$
Remarks about $F_{t,T}$

\[
\mathbb{E}_Q[V_T \mid U_t = u, Y_t^\varepsilon = y, Z_t^\delta = z] = h^{\varepsilon,\delta}(t, u, y, z, T) = \\
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h_0(t, u, z, T) = \exp \left\{ m + (u - m)e^{-\kappa(T-t)} + \frac{\bar{\eta}^2(z)}{4\kappa} \left(1 - e^{-2\kappa(T-t)}\right) \right\},
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\bar{\eta}^2(z) = \langle \eta^2(\cdot, z) \rangle,
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\[
h_{1,0}(t, u, z, T) = g(t, T)V_3(z)\frac{\partial^3 h_0}{\partial u^3}(t, u, z, T),
\]

\[
h_{0,1}(t, u, z, T) = f(t, T)V_1(z)\frac{\partial^3 h_0}{\partial u^3}(t, u, z, T).
\]
Remarks about $F_{t, T}$

We can write

$$h_0(t, u, z, T) = \mathbb{E}[\overline{V}_T \mid \overline{U}_t = u],$$

where

$$\begin{cases} 
\overline{V}_t = e^{\overline{U}_t}, \\
\overline{dU}_t = \kappa(m - \overline{U}_t)dt + \overline{\eta}(z)\overline{dW}_t^{(0)},
\end{cases}$$

$$\overline{\eta}^2(z) = \langle \eta^2(\cdot, z) \rangle.$$
SDE for $F_{t,T}$

Since $F_{t,T}$ is a $\mathbb{Q}$-martingale:

$$dF_{t,T} = \frac{\partial h^{\varepsilon,\delta}}{\partial u}(t, U_t, Y_t^\varepsilon, Z_t^\delta, T)\eta(Y_t^\varepsilon, Z_t^\delta)dW_t^{(0)} +$$

$$+ \frac{1}{\sqrt{\varepsilon}} \frac{\partial h^{\varepsilon,\delta}}{\partial y}(t, U_t, Y_t^\varepsilon, Z_t^\delta, T)\beta(Y_t^\varepsilon)dW_t^{(1)} +$$

$$+ \sqrt{\delta} \frac{\partial h^{\varepsilon,\delta}}{\partial z}(t, U_t, Y_t^\varepsilon, Z_t^\delta, T)g(Z_t^\delta)dW_t^{(2)}.$$
Since $F_{t,T}$ is a $\mathbb{Q}$-martingale:

$$dF_{t,T} = \frac{\partial h^{\varepsilon,\delta}}{\partial u}(t, U_t, Y^\varepsilon_t, Z^\delta_t, T)\eta(Y^\varepsilon_t, Z^\delta_t)dW^{(0)}_t +$$

$$+ \frac{1}{\sqrt{\varepsilon}} \frac{\partial h^{\varepsilon,\delta}}{\partial y}(t, U_t, Y^\varepsilon_t, Z^\delta_t, T)\beta(Y^\varepsilon_t)dW^{(1)}_t +$$

$$+ \sqrt{\delta} \frac{\partial h^{\varepsilon,\delta}}{\partial z}(t, U_t, Y^\varepsilon_t, Z^\delta_t, T)g(Z^\delta_t)dW^{(2)}_t.$$
SDE for $F_{t,T}$

Since $F_{t,T}$ is a $\mathbb{Q}$-martingale:

$$
\begin{align*}
    dF_{t,T} &= \frac{\partial h^{\varepsilon,\delta}}{\partial u}(t, U_t, Y_t^{\varepsilon}, Z_t^{\delta}, T)\eta(Y_t^{\varepsilon}, Z_t^{\delta})dW_t^{(0)} + \\
    &\quad + \frac{1}{\sqrt{\varepsilon}} \frac{\partial h^{\varepsilon,\delta}}{\partial y}(t, U_t, Y_t^{\varepsilon}, Z_t^{\delta}, T)\beta(Y_t^{\varepsilon})dW_t^{(1)} + \\
    &\quad + \sqrt{\delta} \frac{\partial h^{\varepsilon,\delta}}{\partial z}(t, U_t, Y_t^{\varepsilon}, Z_t^{\delta}, T)\eta(Z_t^{\delta})dW_t^{(2)}.
\end{align*}
$$

The coefficients depend on $U_t$ instead of $F_{t,T}$.

We need to invert $h^{\varepsilon,\delta}$ with respect to $u$ and write the expansion of this inverse:

$$
H^{\varepsilon,\delta}(t, \cdot, y, z, T) = (h^{\varepsilon,\delta}(t, \cdot, y, z, T))^{-1}
$$
SDE for $F_{t, T}$

**Lemma**

Since $h_0(t, u, z)$ is invertible, so is $h^{\varepsilon, \delta}$ at least for small $\varepsilon$ and $\delta$. If we choose $H_0, H_{1,0}, H_{0,1}$ to be

(i) $H_0(t, \cdot, z, T) = (h_0(t, \cdot, z, T))^{-1},$

(ii) $H_{1,0}(t, x, z, T) = -\frac{h_{1,0}(t, H_0(t, x, z, T), z, T)}{\frac{\partial h_0}{\partial u}(t, H_0(t, x, z, T), z, T)},$

(iii) $H_{0,1}(t, x, z, T) = -\frac{h_{0,1}(t, H_0(t, x, z, T), z, T)}{\frac{\partial h_0}{\partial u}(t, H_0(t, x, z, T), z, T)},$

we have

$$H^{\varepsilon, \delta}(t, x, y, z, T) = H_0(t, x, z, T) + \sqrt{\varepsilon}H_{1,0}(t, x, z, T) + \sqrt{\delta}H_{0,1}(t, x, z, T) + O(\varepsilon + \delta).$$
SDE for $F_{t,T}$

Then

$$dF_{t,T} = \psi_{1}^{\epsilon,\delta}(t, F_{t,T}, Y_{t}^{\epsilon}, Z_{t}^{\delta}, T)\eta(Y_{t}^{\epsilon}, Z_{t}^{\delta})dW_{t}^{(0)} +$$

$$+ \frac{1}{\sqrt{\epsilon}}\psi_{2}^{\epsilon,\delta}(t, F_{t,T}, Y_{t}^{\epsilon}, Z_{t}^{\delta}, T)\beta(Y_{t}^{\epsilon})dW_{t}^{(1)} +$$

$$+ \sqrt{\delta}\psi_{3}^{\epsilon,\delta}(t, F_{t,T}, Y_{t}^{\epsilon}, Z_{t}^{\delta}, T)g(Z_{t}^{\delta})dW_{t}^{(2)}$$

with

$$\psi_{1}^{\epsilon,\delta}(t, x, y, z, T) := \frac{\partial h^{\epsilon,\delta}}{\partial u}(t, H^{\epsilon,\delta}(t, x, y, z, T), y, z, T),$$

$$\psi_{2}^{\epsilon,\delta}(t, x, y, z, T) := \frac{\partial h^{\epsilon,\delta}}{\partial y}(t, H^{\epsilon,\delta}(t, x, y, z, T), y, z, T),$$

$$\psi_{3}^{\epsilon,\delta}(t, x, y, z, T) := \frac{\partial h^{\epsilon,\delta}}{\partial z}(t, H^{\epsilon,\delta}(t, x, y, z, T), y, z, T),$$
Options on $F_{t,T}$

- Fix a maturity $S < T$ and a non-path dependent payoff $\varphi$.

- The no-arbitrage price of this vanilla European option on $F_{t,T}$ is given by

$$P^{\varepsilon,\delta}(t, x, y, z, T) = \mathbb{E}_Q[e^{-r(S-t)}\varphi(F_{S,T}) \mid F_{t,T} = x, Y_t^\varepsilon = y, Z_t^\delta = z]$$

and then

$$\begin{cases} \mathcal{L}^{\varepsilon,\delta} P^{\varepsilon,\delta}(t, x, y, z, T) = 0, \\ P^{\varepsilon,\delta}(S, x, y, z, T) = \varphi(x), \end{cases}$$

where
PDE for $F_{t,T}$

\[
\mathcal{L}^{\varepsilon,\delta} = \frac{1}{\varepsilon} \left( \mathcal{L}_0 + \frac{1}{2}(\psi_2^{\varepsilon,\delta})^2 \beta^2(y) \frac{\partial^2}{\partial x^2} + \psi_2^{\varepsilon,\delta} \beta^2(y) \frac{\partial^2}{\partial x \partial y} \right) + \\
+ \frac{1}{\sqrt{\varepsilon}} \left( \rho_1 \psi_1^{\varepsilon,\delta} \psi_2^{\varepsilon,\delta} \eta(y,z) \beta(y) \frac{\partial^2}{\partial x^2} + \rho_1 \psi_1^{\varepsilon,\delta} \eta(y,z) \beta(y) \frac{\partial^2}{\partial x \partial y} \right) + \\
+ \frac{\partial}{\partial t} + \frac{1}{2}(\psi_1^{\varepsilon,\delta})^2 \eta^2(y,z) \frac{\partial^2}{\partial x^2} - \mathbf{r} \cdot + \\
+ \sqrt{\delta} \left( \rho_2 \psi_1^{\varepsilon,\delta} \psi_3^{\varepsilon,\delta} \eta(y,z) g(z) \frac{\partial^2}{\partial x^2} + \rho_2 \psi_1^{\varepsilon,\delta} \eta(y,z) g(z) \frac{\partial^2}{\partial x \partial z} \right) + \\
+ \delta \left( \mathcal{M}_2 + \frac{1}{2}(\psi_3^{\varepsilon,\delta})^2 g^2(z) \frac{\partial^2}{\partial x^2} + \psi_3^{\varepsilon,\delta} g^2(z) \frac{\partial^2}{\partial x \partial z} \right) + \\
+ \sqrt{\delta} \left( \rho_{12} \psi_2^{\varepsilon,\delta} \psi_3^{\varepsilon,\delta} \beta(y) g(z) \frac{\partial^2}{\partial x^2} + \\
+ \rho_{12} \psi_3^{\varepsilon,\delta} \beta(y) g(z) \frac{\partial^2}{\partial x \partial y} + \rho_{12} \beta(y) g(z) \frac{\partial^2}{\partial x \partial z} + \rho_{12} \beta(y) g(z) \frac{\partial^2}{\partial y \partial z} \right) \\
\mathcal{L}_0 = \frac{1}{2} \beta^2(y) \frac{\partial^2}{\partial y^2} + \alpha(y) \frac{\partial}{\partial y} \\
\mathcal{M}_2 = \frac{1}{2} g^2(z) \frac{\partial^2}{\partial z^2} + c(z) \frac{\partial}{\partial z}
\]
PDE for $F_{t,T}$

- Denote by $\psi_{k,i,j}$ the term of order $i$ in $\sqrt{\varepsilon}$ and order $j$ in $\sqrt{\delta}$ in the expansion of $\psi_{\varepsilon,\delta}^k$:

$$\psi_{\varepsilon,\delta}^k = \psi_{k,0,0} + \sqrt{\varepsilon}\psi_{k,1,0} + \sqrt{\delta}\psi_{k,0,1} + \sqrt{\varepsilon}\sqrt{\delta}\psi_{k,1,1} + \cdots$$

- For example, we can compute

$$\psi_{1,0,0}(t, x, T) = e^{-\kappa(T-t)}x.$$

- Since $h_0, h_{1,0}$ and $h_{0,1}$ do not depend on $y$,

$$\psi_{2,0,0} = \psi_{2,1,0} = \psi_{2,0,1} = 0.$$

- Notation: $D_k = x^k \frac{\partial}{\partial x^k}$
PDE for $F_{t,T}$

$$\mathcal{L}^{\varepsilon,\delta} = \frac{1}{\varepsilon} \mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}} \mathcal{L}_1 + \mathcal{L}_2 + \sqrt{\varepsilon} \mathcal{L}_3 + \sqrt{\delta} \mathcal{M}_1 + \frac{\sqrt{\delta}}{\varepsilon} \mathcal{M}_3 + \cdots$$

$$\mathcal{L}_0 = \frac{1}{2} \beta^2(y) \frac{\partial^2}{\partial y^2} + \alpha(y) \frac{\partial}{\partial y}$$

$$\mathcal{L}_1 = \rho_1 e^{-\kappa(T-t)} \eta(y, z) \beta(y) D_1 \frac{\partial}{\partial y}$$

$$\mathcal{L}_2 = \frac{\partial}{\partial t} + \frac{1}{2} e^{-2\kappa(T-t)} \eta^2(y, z) D_2 - r \cdot \frac{1}{2} e^{-2\kappa(T-t)} \frac{\partial \phi}{\partial y} \beta^2(y) D_1 \frac{\partial}{\partial y}$$

$$\mathcal{L}_3 = (\psi_{2,3,0} \beta^2(y) + \rho_1 \psi_{1,2,0} \eta(y, z) \beta(y)) \frac{\partial^2}{\partial x \partial y} - \rho_1 \frac{1}{2} e^{-3\kappa(T-t)} \frac{\partial \phi}{\partial y} (y, z) \eta(y, z) \beta(y) D_2$$
Define the Black operator

\[ \mathcal{L}_B(\sigma) = \frac{\partial}{\partial t} + \frac{1}{2} \sigma^2 x^2 \frac{\partial^2}{\partial x^2} - r. \]

and

\[ \sigma(t, y, z, T) = e^{-\kappa(T-t)} \eta(y, z). \]

Then

\[ \mathcal{L}_2 = \mathcal{L}_B(\sigma(t, y, z, T)) - \frac{1}{2} e^{-2\kappa(T-t)} \frac{\partial \phi}{\partial y} \beta^2(y) D_1 \frac{\partial}{\partial y} \]
Carrying on the formal perturbation arguments, we choose $P_0$ to solve

$$
\begin{cases}
\mathcal{L}_B(\bar{\sigma}(t, z, T))P_0 = 0, \\
P_0(S, x, z, T) = \varphi(x)
\end{cases}
$$

and then

$$P_0(t, x, z, T) = P_B(t, x, \bar{\sigma}^2_t, S(z, T))$$

$$\bar{\sigma}^2_{t, S}(z, T) = \frac{1}{S - t} \int_t^S \bar{\sigma}^2(s, z, T)ds = \bar{\eta}^2(z) \left( \frac{e^{-2\kappa(T - S)} - e^{-2\kappa(T - t)}}{2\kappa(S - t)} \right)$$
A Remark about $P_0$

The Feynman-Kac's representation tells us

$$P_0(t, x, z, T) = \mathbb{E}[e^{-r(S-t)}\varphi(F_S, T) \mid F_{t,T} = u],$$

where

$$dF_{t,T} = F_{t,T} e^{-\kappa(T-t)}\eta^2(z)dW^{(0)}_t$$
The first-order correction for $P_0$ is given by the following combination of Greeks of $P_0$:

$$P^\varepsilon_{1,0}(t, x, z, T) = (S - t)V_3^\varepsilon(z)\lambda_3(t, S, T, \kappa)(D_2 + D_1 D_2)P_B(\bar{\sigma}_t, s(z, T)).$$

$$P^\delta_{0,1}(t, x, z, T) = (S - t)V_0^\delta(z)(\lambda_0(t, S, T, \kappa)D_2 + \lambda_1(t, S, T, \kappa)D_1 D_2)P_B(\bar{\sigma}_t, s(z, T)).$$
Accuracy Theorem

Theorem

We assume

(i) *Existence and uniqueness of the SDEs for fixed $(\varepsilon, \delta)$.*

(ii) *The process $Y^1$ with infinitesimal generator $L_0$ has a unique invariant distribution and is mean-reverting.*

(iii) *The function $\eta(y, z)$ is smooth in $z$ and such the solution $\phi$ to the related Poisson equation is at most polynomially growing.*

(iv) *The payoff function $\varphi(x)$ and its derivatives are smooth and bounded.*

Then

\[ P^{\varepsilon,\delta} = P_B(\bar{\sigma}_t, s(z, T)) + P^{\varepsilon}_{1,0} + P^{\delta}_{0,1} + O(\varepsilon + \delta). \]
In order to compute the first order price approximation, we only need the group market parameters \((\kappa, \tilde{\eta}^2(z), V^0_\delta(z), V^3_3(z))\).

Model independence: the approximation is independent of the choice of \(\alpha, \beta, c, g\).

This approximation can be seen as a correction of the Black option price with effective volatility \(\tilde{\sigma}_{t,S}(z, T)\) using some of its Greeks.

Regularization arguments can be used to prove the accuracy of the approximation for non-smooth payoff like the Call option payoff.
The group market parameters can be direct calibrated to liquid options and used for pricing other derivatives to the same level of approximation. Functional Itô Calculus can be used to prove this assertion:

Bruno Dupire, “Functional Itô Calculus” (2009),
http://ssrn.com/abstract=1435551

One should consider the functional space and time derivatives, $\Delta_t$ and $\Delta_x$.

We have the same interpretation for the first-order approximation:

- the zero-order term will be the option price when the volatility is equal to $\eta(z)$;
- the first-order correction will be a combination of (path-dependent) Greeks of the zero-order term with the same parameters $(\kappa, \eta^2(z), V^\delta_0(z), V^e_3(z))$. 
Another approach would be to consider the Taylor expansion of the payoff \( \varphi \) around the zero-order term of the future price approximation, \( h_0 \). The approach we presented today has the following advantages:

(i) Requires less regularity of the payoff function \( \varphi \).

(ii) Allows direct calibration of the group market parameters to call option prices.

(iii) Considers the right underlying asset (the future contract).
Numerical Example - Call Option

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r$</td>
<td>0.01</td>
</tr>
<tr>
<td>$T$</td>
<td>2 year</td>
</tr>
<tr>
<td>$F_{0,T}$</td>
<td>$\in [50, 70]$</td>
</tr>
<tr>
<td>$S$</td>
<td>1 years</td>
</tr>
<tr>
<td>$K$</td>
<td>60</td>
</tr>
<tr>
<td>$\kappa$</td>
<td>0.4</td>
</tr>
<tr>
<td>$\bar{\eta}(z)$</td>
<td>0.2</td>
</tr>
<tr>
<td>$V_3^\varepsilon(z)$</td>
<td>-0.001</td>
</tr>
<tr>
<td>$V_0^\delta(z)$</td>
<td>0.0008</td>
</tr>
</tbody>
</table>

Leading order term - $F_0$

Correction term - $F_{1,0}$ and $F_{0,1}$
The calibration procedure requires only simple regressions (compare with [2])

data considered were Black implied volatilities of call and put options on the crude-oil future contracts on October 16th, 2013. On this day, 533 implied volatilities are available.

organized as follows: for each future contract (i.e. for each maturity $T_i$), there is one option maturity $T_{0ij}$ and 41 strikes $K_{ijl}$.

contractual specifications, the option maturity is roughly one month before the maturity of its underlying future contract (i.e. $T_i \approx T_{0ij} + 30$).

The future prices are shown in Figure 1. (no seasonality for WTI)
The calibration of our model to all the available data is shown in Figure 3 and in Table 1.
We show in Figures 2 and 3 the implied volatility fit for different maturities, where the solid line is the model implied volatility and the circles are the implied volatilities observed in the market. The shortest maturities implied volatility curves are on the leftmost thread and the maturity increases clockwise. The calibrated group market parameters are given in Table 1. It is important to notice that $V_3^\xi(z)$ and $V_0^\delta(z)$ are indeed small and hence these parameters are compatible with our model.
### Calibration example

#### Table: Calibrated Parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value (maturities greater than 90 days)</th>
<th>Value (all)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{\kappa}$</td>
<td>0.1385</td>
<td>0.30853</td>
</tr>
<tr>
<td>$\tilde{\eta}(z)$</td>
<td>0.21967</td>
<td>0.23773</td>
</tr>
<tr>
<td>$V_3^\varepsilon(z)$</td>
<td>-0.00017637</td>
<td>-0.00011823</td>
</tr>
<tr>
<td>$V_0^\delta(z)$</td>
<td>-0.012656</td>
<td>-0.007633</td>
</tr>
</tbody>
</table>
Figure: Future prices on October 16th, 2013.
Figure: Market (circles) and calibrated (solid lines) implied volatilities for options on crude-oil futures with maturity greater than 90 days.
Figure: Market (circles) and calibrated (solid lines) implied volatilities for options on crude-oil futures using all data available.
Conclusions

- We have derived an efficient way to approximate prices of options on futures in the context of exp-OU process with multiscale stochastic volatility.

- In the same lines of the Equity case, the group market parameters can be direct calibrated to liquid options and used for pricing other derivatives to the same level of approximation.


Name at least 4 things all these pics have in common...
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ANSWER!

1. Math Finance Talk
Name at least 4 things all these pics have in common...

ANSWER!

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2. Mostly nice people (Research in Options RiO Meeting)
Name at least 4 things all these pics have in common...

ANSWER!

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2. Mostly nice people (Research in Options RiO Meeting)
3. Nice place in Brazil
Name at least 4 things all these pics have in common...

ANSWER!

1. Math Finance Talk
2. Mostly nice people (Research in Options RiO Meeting)
3. Nice place in Brazil
4. ...
Thank you Jean-Pierre!